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DUVAR

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From a crossed module of associative K-algebras to a strict 2-algebra

Emre OZEL* and Ummahan EGE ARSLAN**

1. Introduction

To describe (pointed, weak) homotopy 2-types of CW-complexes, J. H. C. Whitehead defined crossed modules in [12, 13]. The concept of crossed modules of commutative algebras was introduced to the literature in [5, 6, 8, 10] studies. In [3, 11], the crossed module definition is given for only associative algebras that are not commutative. In [2], some algebraic results for crossed modules of algebras are given.

In [3], the notion of Cat^1-algebra, which is equivalent to the crossed module concept, is defined. In [9], it has been shown that (commutative) Cat^1-algebra is equivalent to the internal category in the category of commutative K-algebras. In [4], a detailed explanation of this equivalence for associative algebra has been given. In [**?**], the concept of categorical R-algebra is given, and the crossed module is associated with this concept.

Respectively, both Akça and Ege Arslan in [1] used the internal category concept, and Özel in [7], by looking at it from a global 2-categorical perspective, constructed a strict, commutative 2-algebra from a crossed module of commutative algebras.

In this study, our aim is to obtain a strict 2-algebra from a crossed module of associative K-algebras, one of the problems in [7]. In particular, we will follow a different path from the construction of the commutative algebra in [7], using the definition of associative action. Then we will show the inverse construction and give the equivalence between the category of crossed modules of associative K-algebras and the category of strict 2-algebras.

2. Preliminaries

In this section, the preliminary definitions required for 2-categorification will be given, and K will be used for the construction, always being a commutative and unitary ring.

Definition 2.1 (Strict 2-algebra)**.** *A strict* 2*-algebra is a strict* 2*-category with a single object whose* 2*-, and* 1*-morphisms sets are associative* K*-algebras. For a more comprehensive definition, see [7]. Let* $\mathfrak{A}$

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be a strict 2*-algebra; this structure is expressed by the following diagram.*

$$\mathfrak{A} = \left(\xymatrix{ A_2 \ar@<0.5ex>[r]^{s_1} \ar@<-0.5ex>[r]_{t_1} & A_1 \ar@/^2pc/[l]^{i_1} \ar@<0.5ex>[r]^{s_0} \ar@<-0.5ex>[r]_{t_0} & A_0 = \{\bullet\} \ar@/^2pc/[l]^{i_0} } \right).$$

Note that a strict 2-algebra $\mathfrak{A}$ is actually a strict 2-algebroid with a single object.

Definition 2.2 (Morphism of strict 2-algebras)**.** *Let $\mathfrak{A}$ and $\mathfrak{A}'$, be strict* 2*-algebras. A strict* 2*-algebra K-homomorphism $\mathfrak{F} = (F_2, F_1, F_0) : \mathfrak{A} \to \mathfrak{A}'$ is strict* 2*-functor given by K-algebra homomorphisms $F_0 : A_0 = \{\bullet\} \to A'_0 = \{\bullet'\}$, $F_1 : A_1 \to A'_1$, and $F_2 : A_2 \to A'_2$ making the following diagram*

$$\begin{array}{c} \mathfrak{A} \\ \Big\downarrow {\scriptstyle \mathfrak{F}=(F_2,F_1,F_0)} \\ \mathfrak{A}' \end{array} \quad = \quad \left(\xymatrix{ A_2 \ar@<0.5ex>[r]^{s_1} \ar@<-0.5ex>[r]_{t_1} \ar[d]_{F_2} & A_1 \ar@/^1pc/[l]^{i_1} \ar@<0.5ex>[r]^{s_0} \ar@<-0.5ex>[r]_{t_0} \ar[d]_{F_1} & A_0 = \{\bullet\} \ar@/^1pc/[l]^{i_0} \ar[d]^{F_0} \\ A'_2 \ar@<0.5ex>[r]^{s'_1} \ar@<-0.5ex>[r]_{t'_1} & A'_1 \ar@/^1pc/[l]^{i'_1} \ar@<0.5ex>[r]^{s'_0} \ar@<-0.5ex>[r]_{t'_0} & A'_0 = \{\bullet'\} \ar@/^1pc/[l]^{i'_0} } \right)$$

commutative.

The category whose objects are strict 2-algebras and whose morphisms are 2-algebra homomorphisms between strict 2-algebras is called the category of strict 2-algebras and is denoted by **2**-$\mathfrak{ALG}$.

Definition 2.3 (Action of associative K-algebras)**.** *Let M and N be associative K-algebras.*

$$\begin{array}{rclcrcl} *_1 : N \times M & \longrightarrow & M & \quad \textit{and} \quad & *_2 : M \times N & \longrightarrow & M \\ (n,m) & \longmapsto & n *_1 m, & & (m,n) & \longmapsto & m *_2 n, \end{array}$$

maps are left and right actions, respectively, of N on M if and only if

1. $n *_1 (m_1 + m_2) = n *_1 m_1 + n *_1 m_2$,
2. $(n_1 + n_2) *_1 m = n_1 *_1 m + n_2 *_1 m$,
3. $n *_1 (m_1 m_2) = (n *_1 m_1) m_2$,
4. $(n_1 n_2) *_1 m = n_1 *_1 (n_2 *_1 m)$,
5. $(m_1 + m_2) *_2 n = m_1 *_2 n + m_2 *_2 n$,
6. $m *_2 (n_1 + n_2) = m *_2 n_1 + m *_2 n_2$,
7. $(m_1 m_2) *_2 n = m_1 (m_2 *_2 n)$,
8. $m *_2 (n_1 n_2) = (m *_2 n_1) *_2 n_2$,
9. $k * (n *_1 m) = (k * n) *_1 m = n *_1 (k * m)$,

10. $k * (m *_2 n) = m *_2 (k * n) = (k * m) *_2 n$,

for all $k \in K$, $n, n_1, n_2 \in N$ *and* $m, m_1, m_2 \in M$.

Remark 2.4 (Associative action)**.** *If* N *has an action on* M *as follows, it is called an associative action*

$$n_1 *_1 (m *_2 n_2) = (n_1 *_1 m) *_2 n_2,$$

for each $n \in N$ *and* $m \in M$.

Definition 2.5 (Crossed module of associative K-algebras)**.** *Let* M *and* N *be associative* K*-algebras. A crossed module of associative* K*-algebras* $A = \left(M \xrightarrow{\partial} N\right)$ *is a* K*-algebra homomorphism* $\partial : M \to N$ *together with an (both left and right)* N*-action on* M *such that;*

$\mathfrak{XMOD}1$ $\partial(n_1 *_1 m_2) = n_1\partial_1(m_2)$ *and* $\partial(m_1 *_2 n_2) = \partial(m_1)n_2$,

$\mathfrak{XMOD}2$ $\partial(m_1) *_1 m_2 = m_1 m_2$ *and* $m_1 *_2 \partial(m_2) = m_1 m_2$,

for each $n \in N$ *and* $m \in M$.

Definition 2.6 (Morphism of crossed module of associative K-algebras)**.** *Let* $A = \left(M \xrightarrow{\partial} N\right)$ *and* $A' = \left(M' \xrightarrow{\partial'} N'\right)$ *be two crossed modules of associative* K*-algebras. An* $f = (f_1, f_0)$ *morphism of crossed modules from* A_1 *to* A'_1 *is illustrated by the following commutative diagram and this diagram preserves the associative* K*-algebras action of* N *on* M*:*

$$\begin{array}{c} A \\ \Big\downarrow {\scriptstyle f=(f_1,f_0)} \\ A' \end{array} = \left(\begin{array}{ccc} M & \xrightarrow{\partial} & N \\ {\scriptstyle f_1}\Big\downarrow & & \Big\downarrow{\scriptstyle f_0} \\ M' & \xrightarrow[\partial']{} & N' \end{array} \right)$$

The category whose objects are crossed modules of associative K-algebras and whose morphisms are crossed module homomorphisms between crossed modules of associative K-algebras is called the category of crossed modules of associative K-algebras and is denoted by $\mathfrak{XMOD}$.

3. Categorification

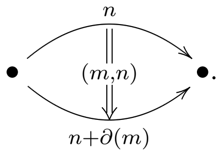
In this section we will obtain a strict 2-algebra from a crossed module of associative K-algebras. Then we will obtain a strict 2-algebra morphism from a crossed module morphism.

The strict 2-algebra $\mathfrak{A}(A)$ obtained from a crossed module of associative K-algebras $A = \left(M \xrightarrow{\partial} N\right)$ is constructed in the following steps:

The set of objects (0-cells): The objects set A_0 of $\mathfrak{A}(A)$ consists of a single object.

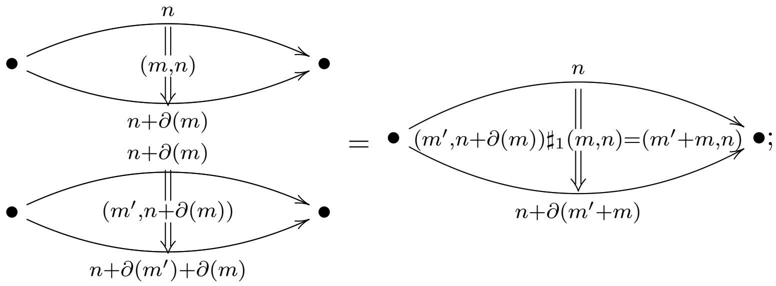
The set of 1-morphisms (1-cells): The 1-morphism set A_1 of $\mathfrak{A}(A)$ is the bottom associative K-algebra N of the crossed module A. Therefore, the composition $\sharp_0$ of 1-morphisms A_1 is the multiplication of N.

The set of 2-morphisms (2-cells): The set of 2-morphisms of $\mathfrak{A}(A)$ is $M \rtimes N$, which is the semi-direct product algebra of associative K-algebras M and N that form the crossed module A. The 2-categorical visualization of any 2-morphism $(m, n) \in M \rtimes N$ is as follows:



The 1-source and 1-target of the 2-morphism (m, n) are $s_1((m, n)) = n$ and $t_1((m, n)) = n + \partial(m)$, respectively.

The vertical composition $\sharp_1$ of 1-compatible 2-morphisms of 1-source and 1-target is expressed by the following diagrams.



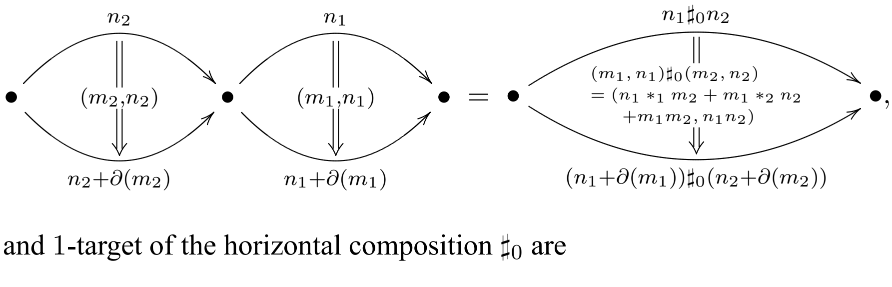
The 1-source and 1-target of $(m', n + \partial(m))\sharp_1(m, n) = (m' + m, n)$ are n and $n + \partial(m') + \partial(m)$, respectively.

Since $M \rtimes N$ is an associative K-algebra, there is an interchange law between the vertical composition and addition.

The horizontal composition $\sharp_0$ of 2-morphisms of $\mathfrak{A}(A)$ is the multiplication of the semi-direct product algebra $M \rtimes N$. The horizontal composition of the 2-morphisms (m_1, n_1) and (m_2, n_2) whose 0-sources and 0-targets are compatible is

$$(m_1, n_1)\sharp_0(m_2, n_2) \quad = \quad (n_1 *_1 m_2 + m_1 *_2 n_2 + m_1m_2, n_1n_2),$$

expressed in the diagram below.



The 1-source and 1-target of the horizontal composition $\sharp_0$ are

$$\begin{aligned} s_1((m_1, n_1)\sharp_0(m_2, n_2)) &= s_1((n_1 *_1 m_2 + m_1 *_2 n_2 + m_1m_2, n_1n_2)) \\ &= n_1n_2 \\ &= n_1\sharp_0 n_2, \end{aligned}$$

and

$$\begin{aligned} t_1((m_1,n_1)\sharp_0(m_2,n_2)) &= t_1((n_1 *_1 m_2 + m_1 *_2 n_2 + m_1m_2, n_1n_2)) \\ &= n_1n_2 + \partial(n_1 *_1 m_2 + m_1 *_2 n_2 + m_1m_2) \\ &= n_1n_2 + \partial(n_1 *_1 m_2) + \partial(m_1 *_2 n_2) + \partial(m_1m_2) \\ &= n_1n_2 + n_1\partial(m_2) + \partial(m_1)n_2 + \partial(m_1)\partial(m_2) \\ &= n_1(n_2 + \partial(m_2)) + \partial(m_1)(n_2 + \partial(m_2)) \\ &= (n_1 + \partial(m_1))\sharp_0(n_2 + \partial(m_2)), \end{aligned}$$

respectively.

Since A is a crossed module over associative K-algebras and N has an associative action on M, the horizontal composition $\sharp_0$ of 2-morphisms is associative. Let's explain this situation in detail.

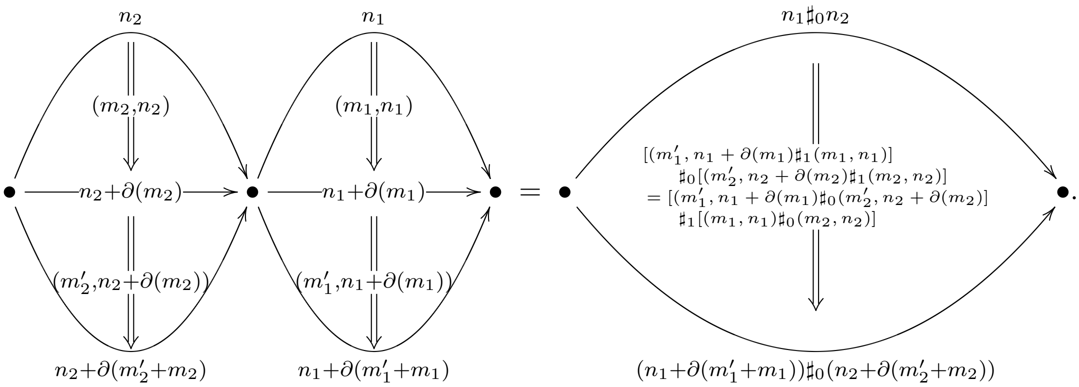
$$\begin{aligned} &(m_1,n_1)\sharp_0\big((m_2,n_2)\sharp_0(m_3,n_3)\big) \\ &\quad = (m_1,n_1)\sharp_0(n_2 *_1 m_3 + m_2 *_2 n_3 + m_2m_3, n_2n_3) \\ &\quad = (n_1 *_1 (n_2 *_1 m_3 + m_2 *_2 n_3 + m_2m_3) + m_1 *_2 (n_2n_3) \\ &\qquad + m_1(n_2 *_1 m_3 + m_2 *_2 n_3 + m_2m_3), n_1(n_2n_3)) \\ &\quad = (n_1 *_1 (n_2 *_1 m_3) + n_1 *_1 (m_2 *_2 n_3) + n_1 *_1 (m_2m_3) \\ &\qquad + (m_1 *_2 n_2) *_2 n_3 + m_1(n_2 *_1 m_3) + m_1(m_2 *_2 n_3) \\ &\qquad + m_1(m_2m_3), n_1(n_2n_3)) \\ &\quad = (n_1 *_1 (n_2 *_1 m_3) + (n_1 *_1 m_2) *_2 n_3 + (m_1 *_2 n_2) *_2 n_3 \\ &\qquad + (m_1m_2) *_2 n_3 + (n_1 *_1 m_2)m_3 + (m_1 *_2 n_2)m_3 \\ &\qquad + (m_1m_2)m_3, (n_1n_2)n_3) \\ &\quad = ((n_1n_2) *_1 m_3 + (n_1 *_1 m_2 + m_1 *_2 n_2 + m_1m_2) *_2 n_3 \\ &\qquad + (n_1 *_1 m_2 + m_1 *_2 n_2 + m_1m_2)m_3, (n_1n_2)n_3) \\ &\quad = (n_1 *_1 m_2 + m_1 *_2 n_2 + m_1m_2, n_1n_2)\sharp_0(m_3,n_3) \\ &\quad = \big((m_1,n_1)\sharp_0(m_2,n_2)\big)\sharp_0(m_3,n_3) \end{aligned}$$

There is also an interchange law between the horizontal composition $\sharp_0$ of 2-morphisms and + binary operation on $M \rtimes N$.

Additionally, there is an interchange law between the vertical $\sharp_1$ and the horizontal $\sharp_0$ compositions of 2-morphisms of $\mathfrak{A}(A)$ as follows:

$$\begin{aligned} &[(m_1', n_1 + \partial(m_1)\sharp_1(m_1,n_1)]\sharp_0[(m_2', n_2 + \partial(m_2)\sharp_1(m_2,n_2)] \\ &\quad = (m_1' + m_1, n_1)\sharp_0(m_2' + m_2, n_2) \\ &\quad = (n_1 *_1 (m_2' + m_2) + (m_1' + m_1) *_2 n_2 + (m_1' + m_1)(m_2' + m_2), n_1n_2) \\ &\quad = (n_1 *_1 m_2' + n_1 *_1 m_2 + m_1' *_2 n_2 + m_1 *_2 n_2 + m_1'm_2' + m_1'm_2 \\ &\qquad + m_1m_2' + m_1m_2, n_1n_2) \\ &\quad = (n_1 *_1 m_2' + \partial(m_1) *_1 m_2' + m_1' *_2 n_2 + m_1' *_2 \partial(m_2) + m_1'm_2' \\ &\qquad + n_1 *_1 m_2 + m_1 *_2 n_2 + m_1m_2, n_1n_2) \\ &\quad = ((n_1 + \partial(m_1) *_1 m_2' + m_1' *_2 (n_2 + \partial(m_2)) + m_1'm_2' + n_1 *_1 m_2 \\ &\qquad + m_1 *_2 n_2 + m_1m_2, n_1n_2) \\ &\quad = ((n_1 + \partial(m_1) *_1 m_2' + m_1' *_2 (n_2 + \partial(m_2)) + m_1'm_2', (n_1 + \partial(m_1)(n_2 + \partial(m_2)) \\ &\qquad \sharp_1(n_1 *_1 m_2 + m_1 *_2 n_2 + m_1m_2, n_1n_2) \\ &\quad = [(m_1', n_1 + \partial(m_1)\sharp_0(m_2', n_2 + \partial(m_2)]\sharp_1[(m_1,n_1)\sharp_0(m_2,n_2)]. \end{aligned}$$

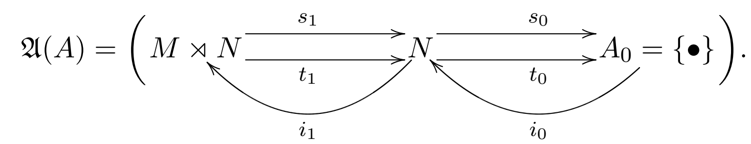
This interchange law is expressed with the categorical diagram as follows:



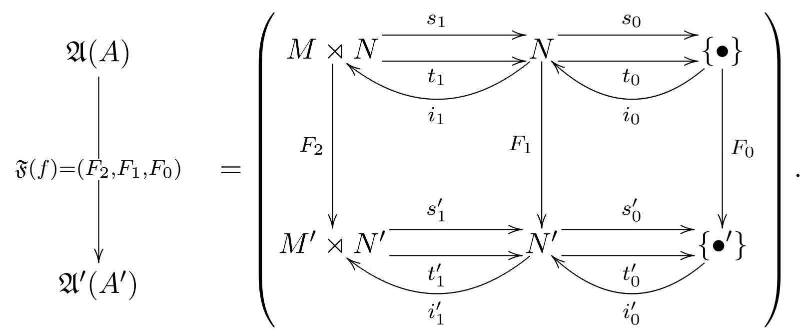
Note that this interchange law is satisfied because of the $\mathfrak{XMOD}2$ axiom in Definition 2.5.

Also, this structure is 2-globular and 2-reflexive; see [7] for details.

The strict 2-algebra $\mathfrak{A}(A)$ constructed from crossed module A of associative K-algebras is expressed by the categorical diagram as follows.



Let A and A' be crossed modules. Let $f(f_1, f_0) : A \to A'$ be a crossed module morphism given in Definition 2.6. Therefore, the strict 2-algebra homomorphism constructed from the crossed module morphism is



This means that,

$$
\begin{aligned}
&F_2((m,n)) = (f_1(m), f_0(n)) \\
&\quad F_1(n) = f_0(n), \quad F_1(n+\partial_1(m)) = f_0(n+\partial_1(m)) = f_0(n) + \partial_1' f_1(m), \\
&\quad F_0(\bullet) = \bullet',
\end{aligned}
$$

for all $n \in N$ and $m \in M$. It is easy to see that F_2, F_1 and F_0 are K-algebra homomorphisms (See, [7] for more detailed explanation.).

Therefore, the following functor can be defined with all the constructions in this study.

$$\mathfrak{XMOD} \xrightarrow{\mathfrak{A}^*=(\mathfrak{A}(-),\mathfrak{F}_1(-))} 2\text{-}\mathfrak{ALG}.$$

4. Conclusion

In this study, we obtained a strict 2-algebra from a crossed module associative K-algebras. The reverse of this construction can also be considered. Similar studies can also be conducted for crossed modules of different algebraic objects.

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Bimultipliers of Crossed Modules of R-Algebroid

Gizem KAHRIMAN

1. Introduction

In the realm of group theory, the interplay between groups and their actions on one another is a subject of profound importance. Central to this discourse is the notion that the action of a group on another group is intricately determined by the automorphism group. This relationship is encapsulated in the form of a homomorphism, mapping the acting group to the automorphism group of the target group. Moreover, any extension of groups also finds its roots in such homomorphisms, further underscoring their significance in understanding the dynamics between groups.

Extending beyond the confines of group theory, similar principles resonate in the domain of algebra, where the action of an algebra on another is closely intertwined with the concept of multiplication algebras. The seminal work of Maclane [1] lays the foundation for this concept, elucidating its pivotal role in algebraic structures. Building upon this framework, Ege and Arvasi [2] introduce actor crossed modules of commutative algebras, leveraging multiplication algebras to generalize aspects from commutative algebras to crossed modules [13], [14].

Within the realm of R-algebroids, a branch of algebraic structures, significant attention has been directed towards their study, notably by Mitchell [3], [4], [5] and Amgott [6]. Mitchell's categorical definition of R-algebroids and Mosa's introduction of crossed modules of R-algeb-
roids serve as pivotal contributions to this field. Notably, the equivalence between crossed modules of R-algebroids and special double algebroids with connections, established by Mosa [7], further enriches our understanding of these structures. Subsequent investigations by Akca and Avcioglu [8], [9], [10], [11], [12] delve deeper into crossed modules of R-algebroids, unraveling intricate connections and properties. By means of algebra action, the 2-crossed module structure is defined [15] and the equivalence of 2-crossed modules to simplicial algebras is shown [16]. There are also studies [17], [18], [19], [20], [21] on 2-crossed modules.

In this paper, we embark on a journey to explore the multifaceted landscape of R-algebroids, with a specific focus on their actions and associated properties. Our endeavor begins with the introduction of the set denoted Bim(M), comprising multipliers of an R-algebroid M. Through a rigorous exposition, we establish that this set itself forms an R-algebroid, aptly termed the multiplication R-algebroid, by defining suitable operations. Leveraging this newfound structure, we define an R-algebroid morphism from an arbitrary algebra to Bim(M), thereby elucidating the mechanism through which actions manifest. Finally, we undertake a comprehensive examination of the properties entailed by this action, shedding light on its intricacies and implications. The concept of bimultipliers of R-Algebroids and their interaction with other R-Algebroids is elucidated in [22]. Lavendhomme and Lucas explore the interplay between bimultiplication algebra and the crossed module structure in their research. In our investigation, we will define bimultipliers of an R-Algebroid crossed module (M, A, η), denoting the set of bimultipliers as $Bim(M, A, \eta)$. Subsequently, we will establish the R-Algebroid nature of this set.

Throughout our discourse, we maintain R as a fixed commutative ring, anchoring our investigations within a well-defined mathematical framework. As we delve deeper into the intricacies of R-algebroids and their actions, we aim to uncover novel insights and forge connections that resonate across various mathematical domains.

Throughout this paper R will be a fixed commutative ring.

1.1. Preliminaries

Most of the following data can be found in [3, 4, 5].

Definition 1.1. *An R-category is defined as a category in which each homset possesses an R-module structure, and the composition is R-bilinear. Consequently, a category earns the designation of an R-category only when it satisfies these conditions.*

Specifically, a small R-category, termed as an R-algebroid, delineates a more specialized class within this framework. This classification is attributed to a category where homsets exhibit an R-module structure, composition is R-bilinear, and additionally, the category is small in size.

Definition 1.2. *An R-linear functor, denoted as an R-functor, denotes a functorial mapping between two R-categories, preserving the R-module structures inherent in their homsets. This functor encapsulates the essence of R-linearity within the categorical framework.*

Moreover, within the realm of R-algebroids, an R-functor between two such structures assumes the appellation of an R-algebroid morphism. This morphism elucidates the preservation of the algebraic structure, including R-linearity and compositionality, between the respective R-algebroids.

Definition 1.3. *Let A be a pre-R-algebroid, and consider the family $I = \{I(x;y) \subseteq A(x;y) : x, y \in A_0\}$ of R-submodules. If $ab, ba' \in I$ for all $b \in I$, $a, a' \in A$ with $ta = sb$, $tb = sa'$, then I is denoted as a two-sided ideal of A.*

Definition 1.4. *Let A and N be two pre-R-algebroids sharing the same object set A_0. Consider a family of maps defined for all $x, y, z \in A_0$ as follows:*

$$\begin{array}{ccc} N(x,y) \times A(y,z) & \longrightarrow & N(x,z) \\ (n,a) & \mapsto & n^a \end{array}$$

is called a right action of A on N if the conditions

1. $n^{a_1+a_2} = n^{a_1} + n^{a_2}$
2. $(n_1 + n_2)^a = n_1^a + n_2^a$
3. $(n^a)^{a'} = n^{aa'}$
4. $(n'n) = n'n^a$
5. $r \centerdot n^a = (r \centerdot n)^a = n^{r \centerdot a}$

and the condition $n^{1_{tn}} = n$, whenever 1_{tn} exists, are satisfied for all $r \in R$, $a, a', a_1, a_2 \in A$, $n, n', n_1, n_2 \in N$ with compatible sources and targets.

In a similar vein, a left action of A on N is established, albeit with a distinction in the side of application. Additionally, if A exhibits both a right and a left action on N, and if the actions conform to the condition $({}^a n)^{a'} = {}^a(n^{a'})$ for all $n \in N$, $a, a' \in A$ with $ta = sn$ and $tn = sa'$, where t denotes the target map and s denotes the source map, then A is termed to possess an associative action on N, or to act associatively on N.

Corollary 1.5. *Given two pre-R-algebroids A and N with the same object set*

i. *if A has a left action on N then ${}^{0_{A(x,sn)}}n = 0_{A(x,tn)}$ and ${}^{-a}n = {}^{a}(-n) = -{}^{a}n$,*

ii. *if A has a right action on N then $n^{0_{A(tn,y)}} = 0_{A(sn,y)}$ and $n^{-a'} = (-n)^{a'} = -n^{a'}$*

for all $n \in N, a, a' \in A, x, y \in A_0$ with $ta = sn, tn = sa'$.

Definition 1.6. *Let M is an R-Algebroid, for all $m, m', m'' \in M$, with $t(m) = s(m')$ and $t(m'') = s(m)$*

$$Ann_M M = \{m \in M : mm' = 0, m''m = 0, m', m'' \in M\}$$

is called Annihilator of M R-Algebroid.

Definition 1.7. *[7] For R-algebroids A and M sharing the same object sets and with A exhibiting an associative action on M, an R-algebroid morphism $\eta : M \to A$ is termed a crossed module of R-algebroids if it satisfies the following conditions:*

$$\begin{array}{ll} CM1. & \eta({}^{a}m) = a\eta(m) \\ & \eta(m^{a'}) = \eta(m)a' \\ CM2. & m^{\eta(m')} = mm' = {}^{\eta(m)} m' \end{array}$$

2. Bimultipliers of an R-algebroid

In this section, we commence our exploration by defining the bimultipliers of an R-algebroid M. Subsequently, we embark on a rigorous proof, establishing that the set of bimultipliers of M indeed forms an R-algebroid, which we aptly term the multiplication R-algebroid. This designation arises from the inherent structure and operations defined on this set, which align with the fundamental principles of R-algebroids.

Definition 2.1. *Let M is an R-Algebroid and $f, g : M \to M$ be an R-Linear mappings with identity on object set satisfying the following equations for $m, m' \in M$ with $t(m) = s(m')$,*

$$\begin{aligned} f(mm') &= mf(m') \\ g(mm') &= g(m)m' \\ f(m)m' &= mg(m') \end{aligned}$$

The pair (f, g) is called bimultipliers of M. Set of all bimultipliers of M are denoted by $Bim(M)$.

Theorem 2.2. *Let $Bim(M)$ be a set of bimultipliers of M. $Bim(M)$ is an R-Algebroid with single object and with the following operations,*

$$\begin{aligned} (f, g) + (f', g') &= (f + f', g + g') \\ (f, g) \circ (f', g') &= (f' \circ f, g \circ g') \\ r \cdot (f, g) &= (r \cdot f, r \cdot g) \end{aligned}$$

Proof.

$$\begin{aligned}
r \cdot ((f,g)+(f',g')) &= r \cdot (f+f', g+g') \\
&= (r \cdot f + r \cdot f', r \cdot g + r \cdot g') \\
&= r \cdot (f,g) + r \cdot (f',g')
\end{aligned}$$

$$\begin{aligned}
(r_1+r_2) \cdot (f,g) &= ((r_1+r_2) \cdot f, (r_1+r_2) \cdot g) \\
&= (r_1 \cdot f + r_2 \cdot f, r_1 \cdot g + r_2 \cdot g) \\
&= (r_1 \cdot f, r_1 \cdot g) + (r_2 \cdot f, r_2 \cdot g) \\
&= r_1 \cdot (f,g) + r_2 \cdot (f,g)
\end{aligned}$$

$$\begin{aligned}
(r_1 r_2) \cdot (f,g) &= (r_1 r_2 \cdot f, r_1 r_2 \cdot g) \\
&= r_1 (r_2 \cdot f, r_2 \cdot g) \\
&= r_1 \cdot (r_2 \cdot (f,g))
\end{aligned}$$

$$\begin{aligned}
r \cdot (f,g) \circ (f',g') &= (r \cdot f, r \cdot g) \circ (f',g') \\
&= ((r \cdot f') \circ f, (r \cdot g) \circ g') \\
&= (r \cdot (f' \circ f), r \cdot (g \circ g')) \\
&= r \cdot (f' \circ f, g \circ g') \\
&= r \cdot ((f,g) \circ (f',g'))
\end{aligned}$$

$$\begin{aligned}
(f,g) \circ r \cdot (f',g') &= (f,g) \circ (r \cdot f', r \cdot g') \\
&= ((r \cdot f') \circ f, g \circ (r \cdot g')) \\
&= (r \cdot (f' \circ f), r \cdot (g \circ g')) \\
&= r \cdot (f' \circ f, g \circ g') \\
&= r \cdot ((f,g) \circ (f' \circ g'))
\end{aligned}$$

□

In the realm of group theory, the characterization of an action is facilitated by the automorphism group. Specifically, for any group A, its action on itself is delineated by a homomorphism $A \to \mathrm{Aut}(A)$. However, in certain algebraic contexts, the mere structure of automorphisms proves insufficient to define an action. Unlike groups, the set of automorphisms of an algebra typically does not form an algebra itself.

In the study conducted by Arvasi and Ege [2], attention is directed towards the case of commutative algebras, where the limitations of the automorphism structure are explored. Furthermore, MacLane [1] delves into the realm of associative algebras, introducing the notion of the bimultiplication algebra $Bim(M)$ associated with an associative algebra M. This concept serves as an alternative to the automorphism group, effectively fulfilling the role of providing an action within the associative algebraic framework.

Definition 2.3. *[22] Let A and M be R-Algebroids with same object we define the set*

$$M \; {}_t\overset{a}{\times}_s \; M = \{(m,m') \in M \times M : t(m) = s(a), t(a) = s(m')\}$$

for an $a \in A$.

Theorem 2.4. *[22] Let A and M be R-Algebroids with same object set and $Ann(M) = 0$ or $M^2 = M$.*

For the maps

$$\begin{aligned} f_a: \quad & M \to M \\ & m \mapsto f_a(m) = m^a \end{aligned}$$

and

$$\begin{aligned} g_a: \quad & M \to M \\ & m' \mapsto g_a(m') =^a m' \end{aligned}$$

for an $a \in A$ *with* $(m, m') \in M \ {}_t\overset{a}{\times}_s \ M$*, let* $(f_a, g_a) \in Bim(M)$*. Then the R-Algebroid morphism*

$$\begin{aligned} \phi: \quad & A \to Bim(M) \\ & a \mapsto \phi(a) = \phi_a = (f_a, g_a) \end{aligned}$$

gives an R-Algebroid action of A on M.

Definition 2.5. *[22] Let A be an R-Algebroid. For an R-Algebroid morphism*

$$\begin{aligned} \phi: A \quad & \to Bim(A) \\ a \quad & \mapsto \phi(a) = (f_a, g_a) \end{aligned}$$

the pair $(f_a, g_a)(a', a'') = (f_a(a'), g_a(a'')) = (a'a, aa'')$ *is called inner bimultipliers of A for* $(a', a'') \in A \ {}_t\overset{a}{\times}_s \ A$*. Set of all bimultipliers of A are denoted by* $I(A)$ *and* $I(A) = Im(\phi)$ *.*

Theorem 2.6. *[22] Let M be an R-Algebroid. The kernel of homomorphism*

$$\begin{aligned} \phi: \quad & M \to Bim(M) \\ & m \mapsto \phi(m) = (f_m, g_m) \end{aligned}$$

is Annihilator of M.

Theorem 2.7. *[22] Let* $I(M)$ *be image of* $\phi : M \to Bim(M)$ *algebroid homomorphism.* $I(M)$ *is ideal of* $Bim(M)$ *.*

Definition 2.8. *[22] Let* $I(M)$ *be ideal of* $Bim(M)$ *algebroid,*

$$O(M) = Bim(M)/I(M)$$

division algebroid is called the outer multiplication of M algebroid and denoted by $O(M)$*.*

Theorem 2.9. *[22] Let M be an R-Algebroid such that* $Ann(M) = 0$ *or* $M^2 = M$ *and*

$$\begin{aligned} \eta: M \quad & \to Bim(M) \\ m \quad & \mapsto \eta(m) = (f_m, g_m) \end{aligned}$$

be an R-Algebroid morphism with the pair $(f_m, g_m)(m', m'') = (f_m(m'), g_m(m'')) = (m'm, mm'')$ *for* $(m', m'') \in M \ {}_t\overset{m}{\times}_s \ M$*. Then* $(M, Bim(M), \eta)$ *is a crossed module.*

Proof. $Bim(M)$ acts on M by

$$\begin{aligned} Bim(M) \times M \quad & \to M \\ ((f', g'), m') \quad & \mapsto (f', g') \cdot m' = g'(m') \end{aligned}$$

and

$$\begin{aligned} M \times Bim(M) &\to M \\ (m'', (f', g')) &\mapsto m'' \cdot (f', g') = f'(m'') \end{aligned}$$

for $(m', m'') \in M \ {}_t\overset{m}{\times}_s \ M$ and

$$\begin{aligned} f'_m : M &\to M \\ m' &\mapsto f'_m(m') = m'm \end{aligned}$$

and

$$\begin{aligned} g'_m : M &\to M \\ m'' &\mapsto g'_m(m'') = mm'' \end{aligned}$$

such that

$$\begin{aligned} \eta : M &\to Bim(M) \\ m &\mapsto \eta(m) = (f'_m, g'_m). \end{aligned}$$

CM1.

$$\begin{aligned} \eta((f', g') \cdot m)(m', m'') &= \eta(g'(m))(m', m'') \\ &= (f'_{g'(m)}, g'_{g'(m)})(m', m'') \\ &= (m'g'(m), g'(m)m'') \\ &= (f'(m')m, g'(mm'')) \\ &= (f'_m(f'(m')), g'(g'_m(m''))) \\ &= (f'_m f', g' g'_m)(m', m'') \\ &= (f', g') \circ (f'_m, g'_m)(m', m'') \end{aligned}$$

then

$$\begin{aligned} \eta((f', g') \cdot m) &= (f', g') \circ (f'_m, g'_m) \\ &= (f', g') \circ \eta(m) \end{aligned}$$

$$\begin{aligned} \eta(m \cdot (f', g'))(m', m'') &= \eta(f'(m))(m', m'') \\ &= (f'_{f'(m)}, g'_{f'(m)})(m', m'') \\ &= (m'f'(m), f'(m)m'') \\ &= (f'(m'm), mg'(m'')) \\ &= (f'(f'_m(m')), g'_m(g'(m''))) \\ &= (f' f'_m, g'_m g')(m', m'') \\ &= (f'_m, g'_m) \circ (f', g')(m', m'') \end{aligned}$$

then

$$\begin{aligned} \eta(m \cdot (f', g')) &= (f'_m, g'_m) \circ (f', g') \\ &= \eta(m) \circ (f', g') \end{aligned}$$

CM2.

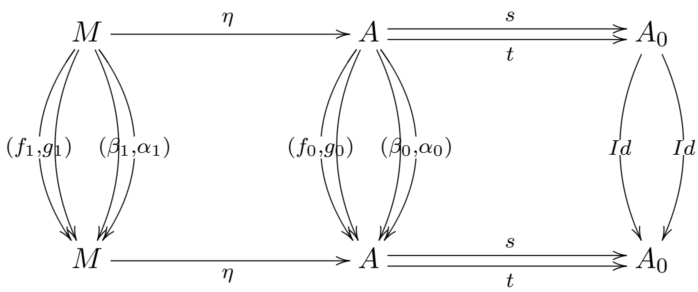
$$\begin{aligned} \eta(m') \cdot m &= (f'_{m'}, g'_{m'}) \\ &= g'_{m'}(m) \\ &= m'm \end{aligned}$$

$$\begin{aligned} m' \cdot \eta(m) &= m' \cdot (f'_m, g'_m) \\ &= f'_m(m') \\ &= m'm \end{aligned}$$

Thus $(M, Bim(M), \eta)$ is a crossed module. $\square$

3. Bimultipliers of Crossed Module of R-algebroid

In this section, the bimultipliers of R-algebroid Crossed Modules will be defined, and it will be shown that the set of bimultipliers is R-algebroid. For (M, A, η) crossed module of R-algebroid



(i) $(f_0, g_0) \in Bim(A)$ for all $a, a' \in A$, with $t(a) = s(a')$,

$$\begin{aligned} f_0(aa') &= af_0(a') \\ g_0(aa') &= g_0(a)a' \\ f_0(a)a' &= ag_0(a') \end{aligned}$$

and $(f_1, g_1) \in Bim(M)$ for all $m, m' \in M$ with $t(m) = s(m')$,

$$\begin{aligned} f_1(mm') &= mf_1(m') \\ g_1(mm') &= g_1(m)m' \\ f_1(m)m' &= mg_1(m') \end{aligned}$$

(ii) For all $m \in M$ and $a \in A$, for all $t(m) = s(a')$ with $t(a) = s(m)$,

$$\begin{aligned} f_1({}^{a}m) &= {}^{a}f_1(m) \\ f_1(m^{a'}) &= m^{f_0(a')} \\ g_1({}^{a}m) &= {}^{g_0(a)}m \\ g_1(m^{a'}) &= g_1(m)^{a'} \\ f_1(m)^{a'} &= m^{g_0(a')} \\ {}^{a}g_1(m) &= {}^{f_0(a)}m \end{aligned}$$

if the conditions are satisfied **f** $= ((f_1, g_1), (f_0, g_0), Id)$ bimultipliers of crossed module of the algeroid is called and denoted by $Bim(M, A, \eta)$. The set $Bim(M, A, \eta)$ forms an R-algebroid structure with the operations defined by

(i) $((f_1, g_1), (f_0, g_0), Id) + ((\alpha_1, \beta_1), (\alpha_0, \beta_0), Id) = ((f_1 + \alpha_1, g_1 + \beta_1), (f_0 + \alpha_0, g_0 + \beta_0), Id)$

(ii) $((f_1, g_1), (f_0, g_0), Id) \circ ((\alpha_1, \beta_1), (\alpha_0, \beta_0), Id) = ((\alpha_1 \circ f_1, g_1 \circ \beta_1), (\alpha_0 \circ f_0, g_0 \circ \beta_0), Id)$

(iii) $r \cdot ((f_1, g_1), (f_0, g_0), Id) = ((r \cdot f_1, r \cdot g_1), (r \cdot f_0, r \cdot g_0), Id)$.

As follows,

$$
\begin{aligned}
& r\cdot[((f_1,g_1),(f_0,g_0),Id)+((\alpha_1,\beta_1),(\alpha_0,\beta_0),Id)] = \\
&= r\cdot[((f_1,g_1)+(\alpha_1,\beta_1)),((f_0,g_0)+(\alpha_0,\beta_0)),Id] \\
&= r\cdot((f_1+\alpha_1,g_1+\beta_1),(f_0+\alpha_0,g_0+\beta_0),Id) \\
&= (r\cdot(f_1+\alpha_1,g_1+\beta_1),r\cdot(f_0+\alpha_0,g_0+\beta_0),(Id)) \\
&= ((r\cdot f_1+r\cdot\alpha_1,r\cdot g_1+r\cdot\beta_1),(r\cdot f_0+r\cdot\alpha_0,r\cdot g_0+r\cdot\beta_0),Id) \\
&= (((r\cdot f_1,r\cdot g_1)+(r\cdot\alpha_1,r\cdot\beta_1)),((r\cdot f_0,r\cdot g_0)+(r\cdot\alpha_0,r\cdot\beta_0)),Id) \\
&= ((r\cdot(f_1,g_1)+r\cdot(\alpha_1,\beta_1)),(r\cdot(f_0,g_0)+r\cdot(\alpha_0,\beta_0)),Id) \\
&= ((r\cdot(f_1,g_1),r\cdot(f_0,g_0),Id)+(r\cdot(\alpha_1,\beta_1),r\cdot(\alpha_0,\beta_0),Id)) \\
&= (r\cdot((f_1,g_1),(f_0,g_0),Id)+r\cdot((\alpha_1,\beta_1),(\alpha_0,\beta_0),Id))
\end{aligned}
$$

$$
\begin{aligned}
(r_1+r_2)\cdot[(f_1,g_1),(f_0,g_0),Id] &= [(r_1+r_2)\cdot(f_1,g_1),(r_1+r_2)\cdot(f_0,g_0),Id] \\
&= [(r_1\cdot(f_1,g_1)+r_2\cdot(f_1,g_1)),(r_1\cdot(f_0,g_0)+r_2\cdot(f_0,g_0)),Id] \\
&= [((r_1\cdot f_1,r_1\cdot g_1)+(r_2\cdot f_1,r_2\cdot g_1)),((r_1\cdot f_0,r_1\cdot g_0) \\
&\quad +(r_2\cdot f_0,r_2\cdot g_0)),Id] \\
&= [(r_1\cdot f_1,r_1\cdot g_1),(r_1\cdot f_0,r_1\cdot g_0),Id] \\
&\quad +[(r_2\cdot f_1,r_2\cdot g_1),(r_2\cdot f_0,r_2\cdot g_0),Id] \\
&= [r_1\cdot(f_1,g_1),r_1\cdot(f_0,g_0),Id]+[r_2\cdot(f_1,g_1),r_2\cdot(f_0,g_0),Id] \\
&= r_1\cdot((f_1,g_1),(f_0,g_0),f)+r_2\cdot((f_1,g_1),(f_0,g_0),Id)
\end{aligned}
$$

$$
\begin{aligned}
(r_1r_2)\cdot[(f_1,g_1),(f_0,g_0),Id] &= [(r_1r_2)\cdot(f_1,g_1),(r_1r_2)\cdot(f_0,g_0),Id] \\
&= [r_1\cdot(r_2\cdot(f_1,g_1)),r_1\cdot(r_2\cdot(f_0,g_0)),Id] \\
&= [r_1\cdot(r_2\cdot f_1,r_2\cdot g_1),r_1\cdot(r_2\cdot f_0,r_2\cdot g_0),Id)] \\
&= r_1\cdot[(r_2\cdot f_1,r_2\cdot g_1),(r_2\cdot f_0,r_2\cdot g_0),Id] \\
&= r_1\cdot[r_2\cdot(f_1,g_1),r_2\cdot(f_0,g_0),Id] \\
&= r_1\cdot[r_2\cdot((f_1,g_1),(f_0,g_0),Id)]
\end{aligned}
$$

$$
\begin{aligned}
& r\cdot[((f_1,g_1),(f_0,g_0),Id)\circ((\alpha_1,\beta_1),(\alpha_0,\beta_0),Id)] = \\
&= r\cdot[((f_1,g_1)\circ(\alpha_1,\beta_1)),((f_0,g_0)\circ(\alpha_0,\beta_0)),Id] \\
&= r\cdot[(\alpha_1\circ f_1,g_1\circ\beta_1),(\alpha_0\circ f_0,g_0\circ\beta_0),Id] \\
&= [r\cdot(\alpha_1\circ f_1,g_1\circ\beta_1),r\cdot(\alpha_0\circ f_0,g_0\circ\beta_0),Id] \\
&= [r\cdot(\alpha_1\circ f_1),r\cdot(g_1\circ\beta_1)],[r\cdot(\alpha_0\circ f_0),r\cdot(g_0\circ\beta_0)],Id \\
&= ([r\cdot(\alpha_1)\circ f_1,g_1\circ r\cdot(\beta_1)],[r\cdot(\alpha_0)\circ f_0,g_0\circ r\cdot(\beta_0)]) \\
&= ([((r\cdot f_1,r\cdot g_1)\circ(f_0,g_0)),((r\cdot f_0,r\cdot g_0)\circ(\alpha_0,\beta_0))],Id \\
&= [((r\cdot f_1,r\cdot g_1),(r\cdot f_0,r\cdot g_0),Id)\circ((\alpha_1,\beta_1),(\alpha_0,\beta_0),Id)] \\
&= (r\cdot(f_1,g_1),r\cdot(f_0,g_0),Id)\circ((\alpha_1,\beta_1),(\alpha_0,\beta_0),Id) \\
&= r\cdot((f_1,g_1),(f_0,g_0),Id)\circ((\alpha_1,\beta_1),(\alpha_0,\beta_0),Id)
\end{aligned}
$$

$$
\begin{aligned}
&r\cdot[((f_1,g_1),(f_0,g_0),Id)\circ((\alpha_1,\beta_1),(\alpha_0,\beta_0),Id)] =\\
&= r\cdot[((f_1,g_1)\circ(\alpha_1,\beta_1)),((f_0,g_0)\circ(\alpha_0,\beta_0)),Id]\\
&= r\cdot[(\alpha_1\circ f_1,g_1\circ\beta_1),(\alpha_0\circ f_0,g_0\circ\beta_0),Id]\\
&= [r\cdot(\alpha_1\circ f_1,g_1\circ\beta_1),r\cdot(\alpha_0\circ f_0,g_0\circ\beta_0),Id]\\
&= [r\cdot(\alpha_1\circ f_1),r\cdot(g_1\circ\beta_1)],[r\cdot(\alpha_0\circ f_0),r\cdot(g_0\circ\beta_0)],Id\\
&= ([\alpha_1\circ r\cdot(f_1),r\cdot(g_1)\circ\beta_1],[\alpha_0\circ r\cdot(f_0),r\cdot(g_0)\circ\beta_0])\\
&= ([((f_1,g_1)\circ(r\cdot f_0,r\cdot g_0)),((f_0,g_0)\circ(r\cdot\alpha_0,r\cdot\beta_0))],Id\\
&= [((f_1,g_1),(f_0,g_0),Id)\circ((r\cdot\alpha_1,r\cdot\beta_1),(r\cdot\alpha_0,r\cdot\beta_0),Id)]\\
&= ((f_1,g_1),(f_0,g_0),Id)\circ(r\cdot(\alpha_1,\beta_1),r\cdot(\alpha_0,\beta_0),Id)\\
&= ((f_1,g_1),(f_0,g_0),Id)\circ r\cdot((\alpha_1,\beta_1),(\alpha_0,\beta_0),Id)
\end{aligned}
$$

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Homotopy of Bimultipliers of Crossed Module of R-algebroids

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1. Introduction

The study of crossed modules of bimultipliers of R-algebroids constitutes a significant and growing direction within algebraic topology and homological algebra. In particular, understanding homotopies between such crossed modules has emerged as an area of considerable mathematical interest, owing to its connections with higher-dimensional categorical structures and homotopical algebra. The purpose of this article is to undertake a comprehensive examination of homotopies of crossed modules of bimultipliers of R-algebroids, presenting refined formulations, structural properties, and recent developments in the field.

We begin by recalling the foundational notions of R-algebroids and crossed modules of bimultipliers, and subsequently establish the homotopical framework in detail. Throughout our exposition, the central ideas are supplemented with illustrative examples and rigorous proofs of the main theorems. This work is intended for advanced graduate students and researchers in algebraic topology, homological algebra, and related disciplines who seek a deeper understanding of the theory of crossed modules of bimultipliers of R-algebroids and its applications.

In group theory, it is a classical observation that the action of one group on another is governed by the automorphism group; such an action corresponds canonically to a homomorphism $A \longrightarrow \mathrm{Aut}(B)$, which also determines extensions of the groups A and B. An analogous phenomenon arises in algebra, where the action of one algebra upon another is encoded in the structure of multiplication algebras—a notion introduced in the seminal work of MacLane [1]. Building on this idea, Ege and Arvasi [2] developed the theory of actor crossed modules of commutative algebras, employing multiplication algebras to transport concepts from commutative algebra to the broader categorical setting of crossed modules.

The theory of R-algebroids has been the subject of extensive investigation, particularly through the contributions of Mitchell [3, 4, 5] and Amgott [6], whose categorical treatment of algebroids has shaped much subsequent research. Mosa [7] introduced crossed modules of R-algebroids and established their equivalence with special double algebroids equipped with connections. Further developments by Akça and Avcıoğlu [8, 9, 10, 11, 12] enriched the structural understanding of crossed modules of R-algebroids and clarified several aspects of their internal behaviour.

The concept of bimultipliers of an R-algebroid and their interaction with other algebroids is investigated in [22]. In particular, Lavendhomme and Lucas examined the interplay between bimultiplication algebras and crossed module structures. In the present work, we introduce the notion of bimultipliers of an R-algebroid crossed module (M, A, η), and denote the corresponding collection by $\mathrm{Bim}(M, A, \eta)$. We subsequently show that this collection naturally inherits the structure of an R-algebroid.

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Furthermore, the formulation of homotopy between crossed module morphisms of R-algebroids, developed by Avcıoğlu [13], forms the conceptual basis for our exploration of homotopies of bimultipliers. We define the homotopy set $U^*(A, M)$ of bimultipliers associated with a crossed module and demonstrate that $U^*(A, M)$ also carries the structure of an R-algebroid. This establishes a homotopical framework within which bimultipliers of crossed modules may be studied systematically.

1.1. Preliminaries

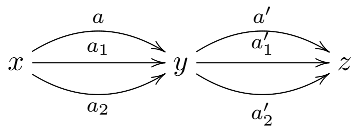
Definition 1.1. *Let R be a commutative ring with identity. For all $x, y, z \in Ob(A)$, $a, a_1, a_2 \in A(x, y)$ and $a', a'_1, a'_2 \in A(y, z)$ and $r, r_1, r_2 \in R$, A category $\mathcal{A}$ of which each homset has an R-module structure*

$$\begin{aligned} R \times A(x,y) &\to A(x,y) \\ (r,a) &\mapsto r \cdot a \end{aligned}$$

$$\begin{array}{lll} i) & r \cdot (a_1 + a_2) & = r \cdot a_1 + r \cdot a_2 \\ ii) & (r_1 + r_2) \cdot a & = r_1 \cdot a + r_2 \cdot a \\ iii) & (r_1 r_2) \cdot a & = r_1 \cdot (r_2 \cdot a) \\ iv) & 1_R \cdot a = a & \end{array}$$

and of which composition is R-bilinear,

$$\begin{aligned} A(x,y) \times A(y,z) &\to A(x,z) \\ (a,a') &\mapsto aa' \end{aligned}$$



$$\begin{array}{lll} i) & (a_1 + a_2)a' & = a_1 a' + a_2 a' \\ ii) & a(a'_1 + a'_2) & = aa'_1 + aa'_2 \\ iii) & r \cdot (aa') & = (r \cdot a)a' = a(r \cdot a') \end{array}$$

is called an R-category.

Definition 1.2. *A small R-category is called an R-Algebroid. An R-Algebroid with a single object corresponds to an associative R-Algebra. Each R-algebroid is pre-R-algebroid.*

Remark 1.3. *In the context of a pre-R-algebroid denoted as $\mathcal{A}$, the subsequent notational conventions are observed:*

1. $Ob(A)$ represents the collection of objects within A, while $Mor(A)$ designates the set of morphisms in A. Additionally, A is described as being over A_0.

2. The functions s and $t : Mor(A) \to A_0$ are referred to as the source and target mappings, respectively. Consequently, for every morphism $a \in Mor(A)$, sa and ta represent the source and target elements of a, respectively. Moreover, a is designated as originating from sa and terminating at ta.

3. For any pair of elements x and y belonging to A_0, the set comprising all morphisms from x to y is symbolized as $A(x, y)$.

4. The identity for the zero morphism within any homset $A(x, y)$ is represented as $0_{A(x,y)}$, or simply as 0 *in cases where clarity is not compromised.*

5. The unit morphism corresponding to any element x in A_0, provided it exists, is symbolized as 1_x, or abbreviated as 1 when context allows for clarity.

6. The notation $a \in A$ succinctly refers to $a \in Mor(A)$, while ab signifies the composition of any pair a and $b \in A$ where $ta = sb$.

Definition 1.4. *A functor that preserves R-linearity between two R-categories is termed an R-functor, while an R-functor operating between two R-algebroids is labeled as an R-algebroid morphism. Furthermore, a mapping between two pre-R-algebroids that adheres to all axioms of an R-functor except for the preservation of identity is termed a pre-R-algebroid morphism.*

It is noteworthy, as per the definition, that every R-algebroid morphism inherently qualifies as a pre-R-algebroid morphism.

Definition 1.5. *Suppose M and A are two pre-R-algebroids sharing the same set of objects, denoted as A_0. A collection of mappings, designated for all elements x, y, and z within A_0, denoted as*

$$\begin{array}{rcl} M(x,y) \times A(y,z) & \rightarrow & M(x,z) \\ (m,a) & \mapsto & m^a \end{array}$$

is termed a right action of A on M if it satisfies certain conditions for all $r \in R$, $a, a', a_1, a_2 \in A$, and $m, m', m_1, m_2 \in M$, where the sources and targets are compatible.

$$\begin{array}{ll} 1) & m^{a_1+a_2} = m^{a_1} + m^{a_2} \\ 2) & (m_1+m_2)^a = m_1^a + m_2^a \\ 3) & (m^a)^{a'} = m^{aa'} \\ 4) & (m'm)^a = m'm^a \\ 5) & r \cdot m^a = (r \cdot m)^a = m^{r \cdot a} \\ 6) & m^{1_y} = m \end{array}$$

and the axiom $m^{1_{tm}} = m$ whenever 1_{tm} exists are satisfied.

Similarly, a left action of A on M is established, albeit with a distinguishing characteristic on one side.

Definition 1.6. *Consider A and M as two pre-R-algebroids sharing an identical set of objects. Should A possess both a right and left action on M, and if $({}^a m)^{a'} =^a (m^{a'})$ holds true for every a and a' in A, as well as for every m in M where $ta = sm$ and $tm = sa'$, then A is deemed to exhibit an associative action on M, or simply, to act on M associatively.*

Definition 1.7. *Consider A as an R-algebroid and M as a pre-R-algebroid sharing the same set of objects. Suppose A has a fixed associative action on M. A pre-R-algebroid morphism $\mu : M \rightarrow A$ is termed a crossed (A-)module of R-algebroids if it adheres to the following axioms:*

$$\begin{array}{ll} CM1) & \mu({}^a m) = a\mu(m) \\ & \mu(m^{a'}) = \mu(m)a' \\ CM2) & m^{\mu(m')} = mm' =^{\mu(m)} m' \end{array}$$

These axioms are satisfied for all $a, a' \in M$ where $ta = sm$ and $tm = sa' = sm'$. It is evident from this definition that if $\mu : M \rightarrow A$ is a crossed module, then it acts as the identity on A_0.

Proposition 1.8. *Suppose $\mu : M \to A$ represents a crossed module of R-algebroids. Then, $(m^a)m' = m({}^a m')$ holds true for every $a \in A$ and $m, m' \in M$ where $tm = sa$ and $ta = sm'$.*

Proof. $(m^a)m' =^{\mu(m^a)} m' =^{(\mu m)a} m' =^{\mu m} ({}^a m') = m({}^a m')$.

It is worth noting from the proposition that if $\mu : M \to A$ stands as a crossed module of R-algebroids, then for any $a \in A$ and $m, m' \in M$ with $tm = sa$ and $ta = sm'$, the notation $m^a m'$ does not lead to ambiguity. □

Definition 1.9. *Given two crossed modules $\mathcal{M} = (\mu : M \to A)$ and $\mathcal{N} = (\eta : N \to B)$ of R-algebroids, a pre-R-algebroid morphism $f_2 : M \to N$ and an R-algebroid morphism $f_1 : A \to B$, if the axioms*

$$\begin{array}{ll} CMM1. & f_2({}^a m) =^{f_1 a} (f_2 m) \text{ and } f_2(m^{a'}) = (f_2 m)^{f_1 a'} \\ CMM2. & \eta f_2 = f_1 \mu \end{array}$$

are satisfied for all $a, a' \in A$ and $m \in M$ with $ta = sm$, $tm = sa'$ then the pair $f = (f_2, f_1)$ is called a crossed module morphism (of R-algebroids) from $\mathcal{M}$ to $\mathcal{N}$ and we write $f : \mathcal{M} \to \mathcal{N}$ to denote it.

Note from the definition that if $f = (f_2, f_1)$ is a crossed module morphism then f_2 and f_1 are equal to each other on object set.

2. Bimultipliers of R-algebroids

Let M be an R-algebroid. Let R-linear transformations $f_1, g_1 : M \to M$ be identity on the object set, if the axioms

$$\begin{array}{ll} (i) & f_1(mm') = mf_1(m') \\ (ii) & g_1(mm') = g_1(m)m' \\ (iii) & f_1(m)m' = mg_1(m') \end{array}$$

are satisfied for all $m, m' \in M$ with $t(m) = s(m')$ then the pair (f_1, g_1) is called a set of all bimultipliers and denoted by $Bim(M)$.

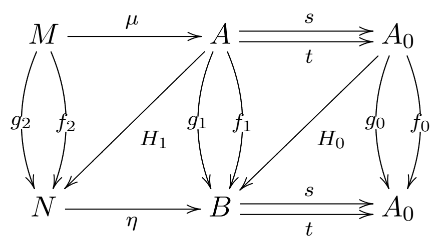
If the axioms

$$\begin{array}{ll} (i) & (f_1, g_1) + (\alpha_1, \beta_1) = (f_1 + \alpha_1, g_1 + \beta_1) \\ (ii) & (f_1, g_1) \circ (\alpha_1, \beta_1) = (\alpha_1 \circ f_1, g_1 \circ \beta_1) \\ (iii) & r(f_1, g_1) = (rf_1, rg_1) \end{array}$$

are satisfied then $Bim(M)$ is R-algebroid. [14]

3. Homotopy Between R-algebroid Crossed Module Morphisms

Let $\mathcal{M} = (M, A, \mu)$ ve $\mathcal{N} = (N, B, \eta)$ crossed modules of R-algebroids and $f = (f_2, f_1, f_0) : \mathcal{M} \to \mathcal{N}$ crossed module morphisms. A mapping $H_0 : A_0 \to B$ and $H_1 : A \to N$ that satisfies the following properties is called an f-derivation on the $H = (H_1, H_0)$ pair. [13]



$*$ For all $x \in A_0$, $t(H_0(x)) = f_0(x)$ and $H_0(x)$ is an isomorphism.

$*$ For all $a \in A$, $s(H_1(a)) = s(H_0(s(a)))$ ve $t(H_1(a)) = f_0(t(a))$.

$*$ For all $x \in A_0$, $H_1(1_x) = 0(= 0_{M(s(H_0(x)),f_0(x)})$.

$*$ For all $a, a' \in A$ ve $t(a) = s(a')$,

$$H_1(aa') = {}^{(H_0(s(a)))f_1(a)(H_0^{-1}(t(a)))}\,(H_1(a')) + (H_1(a))^{f_1(a')} + (H_1(a))({}^{H_0^{-1}(t(a))}(H_1(a')))$$

.

$*$ For all $r \in R$ ve $a, a_1, a_2 \in A$, $s(a_1) = s(a_2), t(a_1) = t(a_2)$

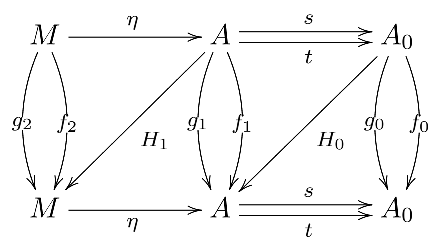
$$\begin{aligned} H_1(a_1 + a_2) &= H_1(a_1) + H_1(a_2) \\ H_1(r \cdot a) &= r \cdot H_1(a). \end{aligned}$$

Let $\mathcal{M} = (M, A, \mu)$ and $\mathcal{N} = (N, B, \eta)$ be crossed module of R-algebroid, $f = (f_2, f_1, f_0) : \mathcal{M} \to \mathcal{N}$ be a crossed module morphism and $H = (H_1, H_0)$ be a f-derivation. The function $g_0 : A_0 \to B_0$ and the R-Algebroid morphisms $g_1 : A \to B$ and $g_2 : M \to N$ are introduced and specified in the following manner.

$$\begin{aligned} g_0(x) &= s(H_0(x)) \\ g_1(a) &= (H_0(s(a))f_1(a) + \eta H_1(a))H_0^{-1}(t(a)) \\ g_2(m) &= ({}^{H_0(s(m))}f_2(m) + H_1(\eta(m)))^{(H_0^{-1}(t(m)))} \end{aligned}$$

This research is centered around algebroid automorphisms. Consequently, the statements defined on the same R-algebroid crossed module are presented as follows.

Let (M, A, η) be crossed module of R-algebroid and $f = (f_0, f_1, f_2)$ be crossed module morphism. If $H_0 : A_0 \to A$ and $H_1 : A \to M$ satisfy the given properties, then the pair $H = (H_0, H_1)$ becomes an f-derivation.



$*$ For all $x \in A_0$, $t(H_0(x)) = f_0(x)$ and $H_0(x)$ is a isomorphism.

$*$ For all $a \in A$, $s(H_1(a)) = s(H_0(s(a)))$ ve $t(H_1(a)) = f_0(t(a))$.

$*$ For all $x \in A_0$, $H_1(1_x) = 0 (= 0_{M(s(H_0(x)), f_0(x)})$.

$*$ For all $a, a' \in A$ ve $t(a) = s(a')$,

$$H_1(aa') = {}^{(H_0(s(a)))f_1(a)(H_0^{-1}(t(a)))}(H_1(a')) + (H_1(a))^{f_1(a')} + (H_1(a))({}^{H_0^{-1}(t(a))}(H_1(a'))).$$

$*$ For all $r \in R$ ve $a, a_1, a_2 \in A$, $s(a_1) = s(a_2), t(a_1) = t(a_2)$

$$\begin{aligned} H_1(a_1 + a_2) &= H_1(a_1) + H_1(a_2) \\ H_1(r \cdot a) &= r \cdot H_1(a). \end{aligned}$$

Accordingly,

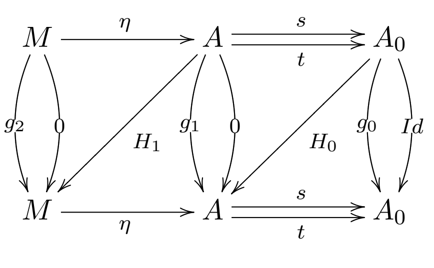
If define $g_0 : A_0 \to A_0$ and $g_1 : A \to A$, $g_2 : M \to M$ R-algebroid morphism

$$\begin{aligned} g_0(x) &= s(H_0(x)) \\ g_1(a) &= (H_0(s(a))f_1(a) + \eta H_1(a))H_0^{-1}(t(a)) \\ g_2(m) &= ({}^{H_0(s(m))}f_2(m) + H_1(\eta(m)))^{(H_0^{-1}(t(m)))} \end{aligned}$$

then $g = (g_0, g_1, g_2)$ is crossed module morphism of R-algebroid and the pair (H_1, H_0) is define a homotopy from f to g. This is expressed as follows $(H_1, H_0) : (f_2, f_1, f_0) \simeq (g_2, g_1, g_0)$.

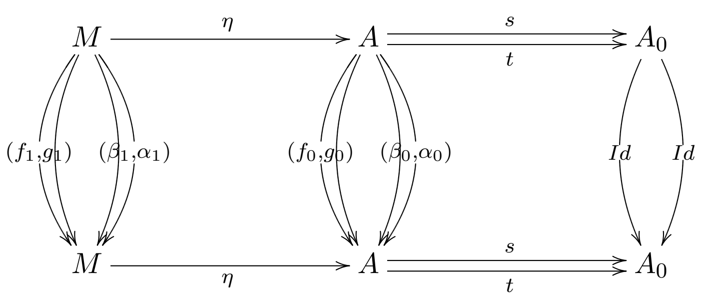
If (g_2, g_1, g_0) is a crossed module morphism $(0, 0, Id) : \mathcal{M} \to \mathcal{M}$ a crossed module morphism homotopic to the zero morphism then $(H_1, H_0) : (0, 0, Id) \simeq (g_2, g_1, g_0)$ and thus

$$\begin{aligned} g_0(x) &= s(H_0(x)) \\ g_1(a) &= \eta H_1(a))H_0^{-1}(t(a)) \\ g_2(m) &= H_1 \eta(m))^{H_0^{-1}(t(m))} \end{aligned}$$



4. Bimultipliers of Crossed Module of R-algebroid

In this section, the bimultipliers of R-algebroid Crossed Modules will be defined, and it will be shown that the set of bimultipliers is R-algebroid. For (M, A, η) crossed module of R-algebroid



(i) $(f_0, g_0) \in Bim(A)$ for all $a, a' \in A$, with $t(a) = s(a')$,

$$\begin{aligned} f_0(aa') &= af_0(a') \\ g_0(aa') &= g_0(a)a' \\ f_0(a)a' &= ag_0(a') \end{aligned}$$

and $(f_1, g_1) \in Bim(M)$ for all $m, m' \in M$ with $t(m) = s(m')$,

$$\begin{aligned} f_1(mm') &= mf_1(m') \\ g_1(mm') &= g_1(m)m' \\ f_1(m)m' &= mg_1(m') \end{aligned}$$

(ii) For all $m \in M$ and $a \in A$, for all $t(m) = s(a')$ with $t(a) = s(m)$,

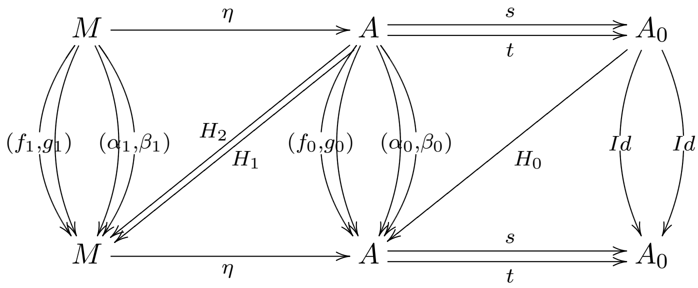
$$\begin{aligned} f_1({}^a m) &= {}^a f_1(m) \\ f_1(m^{a'}) &= m^{f_0(a')} \\ g_1({}^a m) &= {}^{g_0(a)} m \\ g_1(m^{a'}) &= g_1(m)^{a'} \\ f_1(m)^{a'} &= m^{g_0(a')} \\ {}^a g_1(m) &= {}^{f_0(a)} m \end{aligned}$$

if the conditions are satisfied $\mathbf{f} = ((f_1, g_1), (f_0, g_0), Id)$ bimultipliers of crossed module of the algeroid is called and denoted by $Bim(M, A, \eta)$. The set $Bim(M, A, \eta)$ forms an R-algebroid structure with the operations defined by

$$\begin{array}{ll} (i) & ((f_1, g_1), (f_0, g_0), Id) + ((\alpha_1, \beta_1), (\alpha_0, \beta_0), Id) = ((f_1 + \alpha_1, g_1 + \beta_1), (f_0 + \alpha_0, g_0 + \beta_0), Id) \\ (ii) & ((f_1, g_1), (f_0, g_0), Id) \circ ((\alpha_1, \beta_1), (\alpha_0, \beta_0), Id) = ((\alpha_1 \circ f_1, g_1 \circ \beta_1), (\alpha_0 \circ f_0, g_0 \circ \beta_0), Id) \\ (iii) & r \cdot ((f_1, g_1), (f_0, g_0), Id) = ((r \cdot f_1, r \cdot g_1), (r \cdot f_0, r \cdot g_0), Id). \end{array}$$

5. Homotopy of Bimultipliers of Crossed Module of R-algebroid

In this section, we will examine the concept of homotopy for bimultipliers of crossed module of R-algebroid, considering the R-algebroid crossed module morphisms given in 2.1.

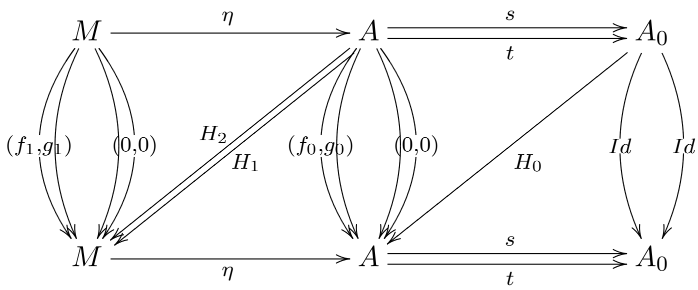


For a crossed module of an R-algebroid (M, A, η), the homotopy of bimultipliers of the crossed module of R-algeroid is given by:

$$\begin{aligned} f(x) &= s(H_0(x)) \\ f_0(a) &= (H_0(s(a))\beta_0(a) + \eta H_2(a))H_0^{-1}(t(a)) \\ g_0(a) &= (H_0(s(a))\alpha_0(a) + \eta H_1(a))H_0^{-1}(t(a)) \\ f_1(m) &= ({}^{H_0(s(m))}\beta_1(m) + H_2\eta(m))^{H_0^{-1}(t(m))} \\ g_1(m) &= ({}^{H_0(s(m))}\alpha_1(m) + H_1\eta(m))^{H_0^{-1}(t(m))} \end{aligned}$$

Where $H_0 : A_0 \to A$ and $H_1, H_2 : A \to M$ are transformations satisfying the equations, and represented by the pair

$$((H_1, H_2), H_0) : ((\alpha_1, \beta_1,), (\alpha_0, \beta_0), Id) \simeq ((f_1, g_1), (f_0, g_0), Id).$$



For a crossed module of R-algebroid (M, A, η), A composition of $((f_1, g_1), (f_0, g_0), Id)$ factors that are homotopic to $((0,0), Id)$ satisfies

$$\begin{array}{rcl} f(x) & = & s(H_0(x)) \\ f_0(a) & = & \eta H_2(a))H_0^{-1}(t(a)) \\ g_0(a) & = & \eta H_1(a))H_0^{-1}(t(a)) \\ f_1(m) & = & H_2\eta(m))^{H_0^{-1}(t(m))} \\ g_1(m) & = & H_1\eta(m))^{H_0^{-1}(t(m))} \end{array}$$

equations and this homotopy is pair of transformations $H_0 : A_0 \to A$ and $H_1, H_2 : A \to M$

$$((H_1, H_2), H_0) : ((0,0), (0,0), Id) \simeq ((f_1, g_1), (f_0, g_0), Id)$$

For $a, a' \in A$ and $g_0 \in Bim(A)$ with $t(a) = s(a')$ and $g_0(aa') = g_0(a)a'$, $t(g_0(a)) = s(a')$. Thus

$$\begin{array}{rcl} t(\eta H_1(a)(H_0^{-1}t(a))) & = & s(a') \\ t(H_0^{-1}t(a)) & = & s(a') \end{array}$$

$H_0t(a) = H_0(y)$ ve $s(H_0^{-1}t(a)) = y$ ve

$$t(H_0^{-1}t(a)) = s(H_0(y)) = s(a') = y$$

$t(H_0(y)) = s(H_0(y))$. Furthermore

$$\begin{array}{rcl} t(H_0t(a)) & = & s(H_0^{-1}t(a)) \\ & = & t(H_0(y)) \\ & = & Id(y) \\ & = & y \end{array}$$

that is,

$$t(H_0^{-1}t(a)) = s(H_0^{-1}t(a))$$

Furthermore

$$\begin{array}{rcl} H_1(aa') & = & H_1(a)^{H_0^{-1}t(a)a'H_0t(a')} \\ H_2(aa') & = & {}^{a}H_2(a) \\ {}^{a}H_1(a') & = & H_2(a)^{H_0^{-1}t(a)a'H_0t(a')}. \end{array}$$

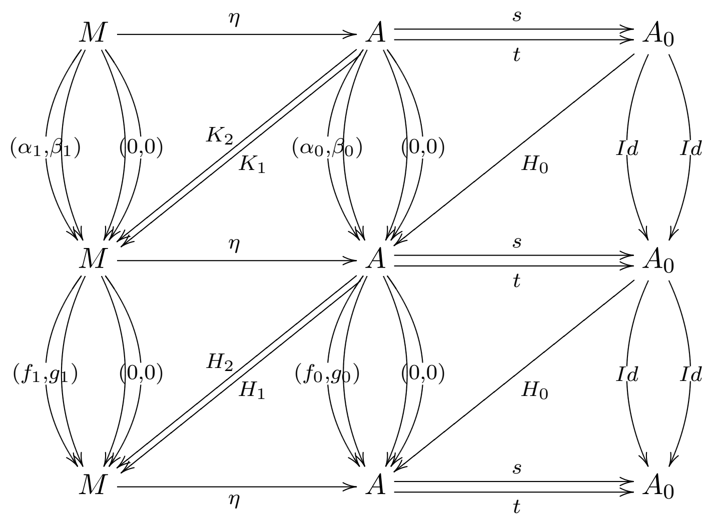
6. U*(A,M)

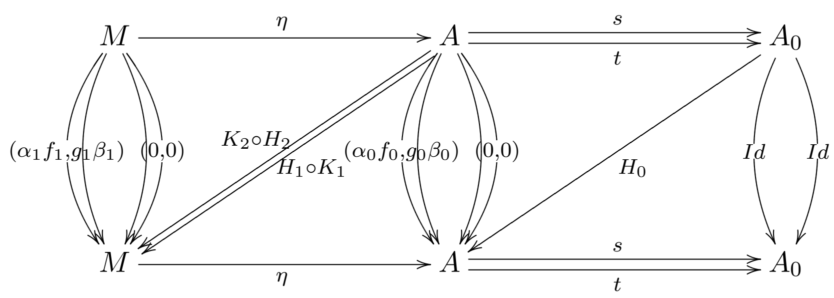
(M, A, η) R-algebroid crossed module and $H_2, H_1 : A \to M$ ve $H_0 : A_0 \to A$ for $a, a' \in A$ with $t(a) = s(a')$,

$$\begin{aligned} H_1(aa') &= H_1(a)^{H_0^{-1}t(a)a'H_0t(a')} \\ H_2(aa') &= {}^{a}H_2(a) \\ {}^{a}H_1(a') &= H_2(a)^{H_0^{-1}t(a)a'H_0t(a')} \end{aligned}$$

provide the conditions $((H_1, H_2), H_0)$ set of the pair denoted by $U(A, M)$. $H_0 : A_0 \to A$ be restricted. That is, for all $((K_1, K_2), K_0) \in U(A, M)$ $K_0 = H_0$ get it. Restricted set $U^*(A, M) = U(A, M)|_{H_0}$ be shown with. The following operations are defined on this set. For $((H_2, H_1), H_0), ((K_2, K_1), H_0) \in U^*(A, M)$, $(K_2 \circ H_2)(a) = K_2\eta H_2(a)H_0^{-1}t(a)$ and $(H_1 \circ K_1)(a) = H_1\eta K_1(a)(H_0^{-1}t(a))$

$$\begin{aligned} (H_2, H_1) + (K_2, K_1) &= (H_2 + K_2, H_1 + K_1) \\ r \cdot (H_2, H_1) &= (r \cdot H_2, r \cdot H_1) \\ (H_2, H_1) \circ (K_2, K_1) &= (K_2 \circ H_2, H_1 \circ K_1). \end{aligned}$$





Proposition 6.1. *Let (M, A, η) be a crossed module of R-algebroids. For any $((H_2, H_1), H_0) \in U^*(A, M)$,*

define

$$\mathrm{Id}(x) = s(H_0(x)),$$

$$f_0(a) = \eta(H_2(a))\, H_0^{-1}(t(a)),$$

$$g_0(a) = \eta(H_1(a))\, H_0^{-1}(t(a)),$$

$$f_1(m) = (H_2(\eta(m)))^{H_0^{-1}(t(m))},$$

$$g_1(m) = (H_1(\eta(m)))^{H_0^{-1}(t(m))}.$$

Then $(f_1, g_1) \in \mathrm{Bim}(M)$ *and* $(f_0, g_0) \in \mathrm{Bim}(A)$.

Proof.

$$\begin{aligned} f_1(mm') &= H_2\eta(mm')^{H_0^{-1}t(mm')} \\ &= H_2(\eta(m)\eta(m'))^{H_0^{-1}t(m')} \\ &= ({}^{\eta(m)}H_2\eta(m'))^{H_0^{-1}t(m')} \\ &= mH_2\eta(m')^{H_0^{-1}t(m')} \\ &= mf_1(m') \end{aligned}$$

$$\begin{aligned} g_1(mm') &= H_1\eta(mm')^{H_0^{-1}t(mm')} \\ &= H_1(\eta(m)\eta(m'))^{H_0^{-1}t(m')} \\ &= H_1\eta(m)^{H_0^{-1}t(\eta(m))\eta(m')H_0t(\eta(m'))H_0^{-1}t(m')} \\ &= H_1\eta(m)^{H_0^{-1}t(\eta(m)\eta(m'))} \\ &= H_1\eta(m)^{H_0^{-1}t(m)\eta(m')} \\ &= H_1\eta(m)^{H_0^{-1}t(m)}m' \\ &= g_1(m)m' \end{aligned}$$

$$\begin{aligned} f_0(aa') &= \eta(H_2(aa'))H_0^{-1}t(aa') \\ &= \eta({}^aH_2(a')H_0^{-1}t(a')) \\ &= a\eta(H_2(a')H_0^{-1}t(a')) \\ &= af_0(a') \end{aligned}$$

$$\begin{aligned} g_0(aa') &= \eta(H_1(aa'))H_0^{-1}t(aa') \\ &= \eta(H_1(a)^{H_0^{-1}t(a)a'H_0t(a')})H_0^{-1}t(a') \\ &= \eta(H_1(a))H_0^{-1}t(a)a'H_0t(a')H_0^{-1}t(a') \\ &= g_0(a)a' \end{aligned}$$

$t(a) = s(m)$ ve $t(m) = s(a')$

$$\begin{aligned}
f_1({}^a m) &= H_2\eta({}^a m)^{H_0^{-1}t({}^a m)} \\
&= H_2(a\eta(m))^{H_0^{-1}t(m)} \\
&= {}^a H_2(\eta(m))^{H_0^{-1}t(m)} \\
&= {}^a f_1(m)
\end{aligned}$$

$$\begin{aligned}
f_1(m^{a'}) &= H_2\eta(m^{a'})^{H_0^{-1}t(m^{a'})} \\
&= H_2(\eta(m)a')^{H_0^{-1}t(a')} \\
&= {}^{\eta(m)}H_2(a')^{H_0^{-1}t(a')} \\
&= mH_2(a')^{H_0^{-1}t(a')} \\
&= m^{\eta H_2(a')H_0^{-1}t(a')} \\
&= m^{f_0(a')}
\end{aligned}$$

$$\begin{aligned}
g_1(m^{a'}) &= H_1\eta(m^{a'})^{H_0^{-1}t(m^{a'})} \\
&= H_1(\eta(m)a')^{H_0^{-1}t(a')} \\
&= (H_1(\eta(m))^{H_0^{-1}t(\eta(m))a'H_0t(a')})^{H^{-1}t(a')} \\
&= H_1(\eta(m))^{H_0^{-1}t(\eta(m))a'} \\
&= H_1(\eta(m))^{H_0^{-1}t(m)a'} \\
&= g_1(m)^{a'}
\end{aligned}$$

$$\begin{aligned}
g_1({}^a m) &= H_1\eta({}^a m)^{H_0^{-1}t({}^a m)} \\
&= H_1(a\eta(m))^{H_0^{-1}t(m)} \\
&= H_1(a)^{H_0^{-1}t(a)\eta(m)H_0t(\eta(m))H_0^{-1}t(m)} \\
&= H_1(a)^{H_0^{-1}t(a)\eta(m)} \\
&= H_1(a)^{\eta({}^{H_0^{-1}t(a)}m)} \\
&= {}^{H_1(a)H_0^{-1}t(a)}m \\
&= {}^{\eta H_1(a)}({}^{H_0^{-1}t(a)}m) \\
&= {}^{\eta H_1(a)H_0^{-1}t(a)}m \\
&= {}^{g_0(a)}m
\end{aligned}$$

$$
\begin{aligned}
{}^{a}g_1(m) &= ({}^{a}H_1\eta(m))^{H_0^{-1}t(m)} \\
&= H_2(a)^{H_0^{-1}t(a)\eta(m)H_0t(\eta(m))H_0^{-1}t(m)} \\
&= H_2(a)^{H_0^{-1}t(a)\eta(m)} \\
&= H_2(a)^{\eta({}^{H_0^{-1}t(a)}m)} \\
&= {}^{H_2(a)H_0^{-1}t(a)}m \\
&= {}^{\eta H_2(a)}({}^{H_0^{-1}t(a)}m) \\
&= {}^{\eta H_2(a)H_0^{-1}t(a)}m \\
&= {}^{f_0(a)}m
\end{aligned}
$$

$$
\begin{aligned}
f_1(m)^{a'} &= H_2\eta(m)^{H_0^{-1}t(m)a'} \\
&= H_2\eta(m)^{H_0^{-1}t(m)a'H_0t(a')H_0^{-1}t(a')} \\
&= ({}^{\eta(m)}H_1(a'))^{H_0^{-1}t(a')} \\
&= (mH_1(a'))^{H_0^{-1}t(a')} \\
&= (m^{\eta H_1(a')})^{H_0^{-1}t(a')} \\
&= m^{\eta H_1(a')H_0^{-1}t(a')} \\
&= m^{g_0(a')}.
\end{aligned}
$$

□

► In this paper $((H_2, H_1), H_0) \in U^*(A, M)$ will be displayed briefly $(H_2, H_1) \in U^*(A, M)$.

Proposition 6.2. *The set $U^*(A, M)$ creates an R-algebraid structure with the operations defined on it.*

Proof. $(H_1, H_2), (K_1, K_2) \in U(M, A)$ için,

$$
\begin{aligned}
(H_1 \circ K_1)(aa') &= H_1(\eta K_1(aa')(K_0^{-1}t(aa')) \\
&= H_1(\eta(K_1(a)^{K_0^{-1}t(a)a'K_0t(a')}))K_0^{-1}t(aa') \\
&= H_1(\eta(K_1(a))K_0^{-1}t(a)a')K_0t(a')K_0^{-1}t(a')
\end{aligned}
$$

$$
\begin{aligned}
&(H_1 \circ K_1)(a)^{(H_0\circ K_0)^{-1}t(a)a'(H_0\circ K_0)t(a')} = \\
&= H_1(\eta K_1(a)K_0^{-1}t(a))^{(H_0^{-1}t(K_0))^{-1}t(a)a'(H_0^{-1}t(K_0))^{-1}t(a')} \\
&= H_1(\eta K_1(a)K_0^{-1}t(a))H_0^{-1}t(K_0^{-1}t(a))a'H_0^{-1}t(a')
\end{aligned}
$$

$$
\begin{aligned}
(H_2 \circ K_2)(aa') &= H_2(\eta K_2(aa')(K_0^{-1}t(aa')) \\
&= H_2(\eta({}^aK_2(a'))(K_0^{-1}t(a')) \\
&= H_2(a\eta K_2(a'))(K_0^{-1}t(a')) \\
&= {}^aH_2\eta K_2(a')(K_0^{-1}t(a')) \\
&= {}^a(H_2 \circ K_2)(a')
\end{aligned}
$$

$$
\begin{aligned}
R \times U^*(A,M) &\rightarrow U^*(A,M) \\
(r,(H_2,H_1)) &\mapsto (r \cdot H_2, r \cdot H_1)
\end{aligned}
$$

olmak üzere

$$
\begin{aligned}
r \cdot [(H_2,H_1) + (K_2,K_1)(a)] &= r \cdot [(H_2+K_2, H_1+K_1)(a)] \\
&= r \cdot [(H_2+K_2)(a), (H_1+K_1)(a)] \\
&= [r \cdot (H_2+K_2)(a), r \cdot (H_1+K_1)(a)] \\
&= [(r \cdot H_2 + r \cdot K_2)(a), (r \cdot H_1 + r \cdot K_1)(a)] \\
&= (r \cdot H_2, r \cdot H_1)(a) + (r \cdot K_2, r \cdot K_1)(a) \\
&= [r \cdot (H_2,H_1) + r \cdot (K_2,K_1)](a)
\end{aligned}
$$

$$
\begin{aligned}
(r_1+r_2) \cdot (H_2,H_1)(a) &= [(r_1+r_2) \cdot H_2(a), (r_1+r_2) \cdot H_1(a)] \\
&= [r_1 \cdot H_2(a) + r_2 \cdot H_2(a), r_1 \cdot H_1(a) + r_2 \cdot H_1(a)] \\
&= (r_1 \cdot H_2(a), r_2 \cdot H_2(a)) + (r_2 \cdot (H_2,H_1)(a)) \\
&= r_1 \cdot (H_2,H_1)(a) + r_2 \cdot (H_2,H_1)(a)
\end{aligned}
$$

$$
\begin{aligned}
(r_1r_2) \cdot (H_2,H_1)(a) &= r_1 \cdot (r_2 \cdot H_2(a), r_2 \cdot H_1(a)) \\
&= (r_1r_2 \cdot H_2(a), r_1r_2 \cdot H_1(a)) \\
&= r_1 \cdot (r_2 \cdot (H_2,H_1))(a)
\end{aligned}
$$

$$
\begin{aligned}
r \cdot ((H_2,H_1) \circ (K_2,K_1))(a) &= (r \cdot (K_2 \circ H_2), r \cdot (H_1 \circ K_1))(a) \\
&= (r \cdot (K_2\eta H_2(a)H_0^{-1}t(a)), r \cdot (H_1\eta K_1(a)H_0^{-1}t(a))) \\
&= (K_2\eta H_2(r \cdot a)H_0^{-1}t(r \cdot a), H_1\eta K_1(r \cdot a)H_0^{-1}t(r \cdot a)) \\
&= (K_2 \circ H_2, H_1 \circ K_1)(r \cdot a) \\
&= (r \cdot K_2 \circ H_2, r \cdot H_1 \circ K_1)(a) \\
&= (r \cdot H_2, r \cdot H_1) \circ (K_2,K_1)(a) \\
&= (r \cdot (H_2,H_1) \circ (K_2,K_1))(a)
\end{aligned}
$$

. □

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From Simplicial R-Algebroids to Crossed Squares of R-Algebroids

Isinsu DOGANAY YALGIN* and Ummahan EGE ARSLAN**

1. Introduction

Whitehead introduced crossed modules of groups for the first time in [1, 2]. Group crossed modules are equivalent to simplicial groups with Moore complex of length one [3] and similarly for groupoid crossed modules [4]. Conduche addressed the idea of a group 2-crossed module and shown in [3] that the category of group 2-crossed modules is equal to the category of simplicial groups with a two-length Moore complex. Arvasi and Ulualan investigated the relationships between simplicial groups with a length of two Moore complex, crossed squares, quadratic modules, and 2-crossed modules in [18]. The definitions of algebra crossed and 2-crossed modules [5, 8] are similar to those of the group case, actions by the automorphisms is replaced by the actions by the multipliers.

Algebra 2-crossed modules and simplicial algebras are closely related, just like in the group case [3, 4, 9]. A 2-crossed module can be obtained from a simplicial algebra if it has a Moore complex of length two. Equivalence from category of simplicial algebra with a two-length Moore complex to category algebra crossed module is given in [8, 10, 11]. Also in [19, 20] Akca and Pak worked on the pseudo simplicial groups. Moreover in [21], the higher order Peiffer elements in simplicial Lie algebras are examined. The homotopy theory of 2 -crossed modules of commutative algebras studied in [22]. Then in [23], the concept of a 2 -fold homotopy between a pair of 1 -fold homotopies connecting 2 -crossed module morphisims was defined. As a more broadly, Mitchell in [12, 14] and Amgott in [15] specifically studied R-algebroids, where R is a commutative ring. R-algebroids were defined categorically by Mitchell (see Definition 1). Later, Mosa introduced crossed modules of R-algebroids as a generalization of crossed modules of associative R-algebras and demonstrated in his thesis [16] that they are equivalent to special double R-algebroids with connections. Additionally, it was mentioned in [17] that there was a close relationship between the category of simplicial R-algebroids with the length one Moore complex and the internal categories in the category of R-algebroids. Subsequent investigations by Akca and Avcıoglu [24, 25, 26, 27, 28] delve deeper into crossed modules of R-algebroids, unraveling intricate connections and properties. Guin- Waléry and Loday defined crossed squares in [30] as an algebraic model for homotopy 3-type connected spaces. Thus crossed squares model homotopy types in dimensions bigger than 3. Later Ellis defined the commutative algebra version of crossed squares in [29]. In this work we introduce R-algebroid version of crossed square. Then we construct a functor from the category of simplicial R-algebroids with Moore complexof length two to the category of crossed squares of R-algebroids.

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This chapter is produced from Ph.D Thesis [31]

1.1. Preliminaries

Most of the following data, can be found in [12, 13, 14, 15, 16].

Definition 1.1. *Let R be a commutative ring. An R-category is a category where composition is R-bilinear and all homsets possess R-module structures. This framework enables the exploration of categorical concepts and constructions within the realm of R-modules, offering a robust foundation for algebraic and categorical inquiries.*

Definition 1.2. *An R-algebroid is a small R-category. R- algebroids can be non identity. A set of functions $s, t : Mor(U) \to Ob(U)$, the source and target functions, respectively, and an object set $Ob(U) = U_0$, a morphism set $Mor(U)$, are included with an R-algebroid U. A single object R-algebroid corresponds to an associative R-algebra.*

Let U and V be R-algebroids and $U_0 = V_0$, if the family of maps

$$\begin{array}{rcl} V(a,b)\times U(b,c) & \to & V(a,c) \\ (v,u) & \mapsto & v^u \end{array} \tag{1}$$

satisfies the following conditions

$$\begin{array}{ll} 1) & v^{u_1+u_2} = v^{u_1} + v^{u_2} \\ 2) & (v_1+v_2)^u = v_1^u + v_2^u \\ 3) & (v^u)^{u'} = v^{uu'} \\ 4) & (v'v)^u = v'v^u \\ 5) & r\cdot v^u = (r\cdot v)^u = v^{r\cdot u} \\ 6) & v^{1_{tv}} = v \end{array} \tag{2}$$

for all $a, b, c \in U_0$ and $u, u', u_1, u_2 \in Mor(U)$,$v, v', v_1, v_2 \in Mor(V)$ such that $t(v') = s(v), t(u) = s(u')$,$t(v) = t(v_1) = t(v_2) = s(u) = s(u_1) = s(u_2)$, $r \in R$ it is called the right action of U on V.

The left action of U on V similarly defined. While U has right and left action on V if the condition

$$({}^{u}v)^{u'} = {}^{u}(v^{u'})$$

is satisfied for all $d, a, b, c \in U_0, v \in V(a, b), u \in U(d, a)$ and $u' \in U(b, c)$ then U has an associative action on V.

Definition 1.3. *An R-functor is an R-linear functor between two R-categories, and an R-algebroid morphism is an R-functor between two R-algebroids.*

In category $Alg(R)$, all R-algebroids and their morphisms are included.

Definition 1.4. *Let R is an commutative ring U and V be two R-algebroids of the same object set U_0 and V has an associative action on U . For the set $U \rtimes V = \{(u, v)|u \in U, v \in V\}$, if the following* *conditions are satisfied*

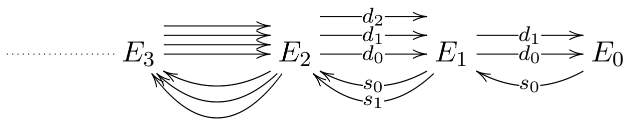
$$
\begin{array}{llcl}
1) & (u,v)+(u',v') & = & (u+u',v+v') \\
2) & R\times(U\rtimes V) & \to & U\rtimes V \\
 & (r,(u,v)) & & {}^{r}(u,v)=({}^{r}u,{}^{r}v) \\
3) & ((u,v),(u'',v'')) & = & (uu''+u^{v''}+{}^{v}u'',vv'')
\end{array}
\tag{3}
$$

$U\rtimes V$ is an R-algebroid, where for all $(u,v)\in U\rtimes V$ and $r\in R$, $s(u,v)=s(u)=s(v), t(u,v)=t(u)=t(v)$, $(u,v),(u',v'),(u'',v'')\in U\rtimes V$, $s(u,v)=s(u',v'), t(u,v)=t(u',v'), t(u,v)=s(u'',v'')$. This R-algebroid is called the semi-direct product R-algebroid of U and V .

Definition 1.5. *A simplicial R-algebroid is a sequence of R-algebroids $E=\{E_0,E_1,...,E_n,...\}$ together with homomorphisms $d_i^n:E_n\to E_{n-1}$ $(0\leq i\leq n\neq 0)$ and $s_j^n:E_n\to E_{n+1}$ $(0\leq j\leq n)$ for each $(0\leq i\leq n\neq 0)$ such that identity on object set, this homomorphisms are requared to satisfy the simplicial identities*

$$
\begin{array}{lll}
1) & d_i^{n-1}d_j^n=d_{j-1}^{n-1}d_j^n & ,\ 0\leq i<j\leq n \\
2) & s_i^{n+1}s_j^n=s_{j+1}^{n+1}s_i^n & ,\ 0\leq i\leq j\leq n \\
3) & d_i^{n+1}s_j^n=s_{j-1}^{n-1}d_i^n & ,\ 0\leq i<j\leq n \\
4) & d_i^{n+1}s_j^n=Id & ,\ i=j \text{ or } i=j+1 \\
5) & d_i^{n+1}s_j^n=s_j^{n-1}d_{i-1}^n & ,\ 0\leq j<i-1\leq n
\end{array}
\tag{4}
$$

We denote this simplicial R-algebroid with $\boldsymbol{E}=(E_n,d_i^n,s_j^n)$.

$\boldsymbol{E}$ = 

*Let $\boldsymbol{E}=(E_n,d_i^n,s_j^n)$ and $\boldsymbol{F}=(F_n,\delta_i^n,\sigma_j^n)$ be R-algebroids. A simplicial map $\boldsymbol{f}=\{f_n:n\in\mathbb{N}\}:\boldsymbol{E}\to\boldsymbol{F}$ is a family of homomorphisms $f_n=E_n\to F_n$ satisfying $\delta_i^nf_n=f_{n-1}d_i^n$ and $f_ns_j^{n-1}=\partial_j^{n-1}f_{n-1}$ for all $n\in\mathbb{N}$. We have thus defined category of simplicial R-algebroids, which we will denote by **Simp.R-Alg.**.*

Let $\boldsymbol{E}$ be a simplicial R-algebroid. The Moore complex (NE,∂) of $\boldsymbol{E}$ is the chain complex defined by $NE_n=\bigcap\limits_{i=0}^{n-1}kerd_i^n$ with $\partial_n:NE_n\to NE_{n-1}$ induced from d_n^n by restriction.

$$
...\to NE_2\xrightarrow{d_2^2}NE_1\xrightarrow{d_1^1}E_0=E_0
$$

*We say that the Moore complex (NE,∂) of $\boldsymbol{E}$ is of length k if $NE_n=0$ for all $n\geq k+1$. We denote category of simplicial R-algebroids with Moore complex of lenght k by **Simp.R-Alg.**$_{\leq k}$.*

2. Crossed Squares of R-algebroids

Guin- Waléry and Loday defined crossed squares in [30] as an algebraic model for homotopy 3-type connected spaces. Thus crossed squares model homotopy types in dimensions bigger than 3. Later

Ellis defined the commutative algebra version of crossed squares in [29]. In this section we introduce R-algebroid version of crossed square.

Definition 2.1. *A crossed square is a commutative square of R-algebroids with the same object set* M_0

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ N & \xrightarrow[\upsilon]{} & P \end{array}$$

together with associative actions P *on* L, M, N *and a function* $h : M \times N \longrightarrow L$ *identity on* M_0 *.Let* M *and* N *act on* M, N *and* L *via* P*. The structure must satisfy the following axioms*

CS 1) λ *and* λ' *preserve the action of* P*, and* $\lambda, \lambda', \mu, \upsilon$ *and* $\upsilon\lambda' = \mu\lambda$ *are crossed modules.*

CS 2) $h(m + m_1, n) = h(m, n) + h(m_1 + n), \;\; h(m, n + n_1) = h(m, n) + h(m, n_1),$

CS 3) $r \centerdot h(m, n) = h(r \centerdot m, n) = h(m, r \centerdot n), \;\; (r \in R)$

CS 4) ${}^{p}h(m, n) = h({}^{p}m, n),$ *and* $h(m, n)^{p'} = h(m, n^{p'})$

CS 5) $h(m'm, n) = {}^{m'}h(m, n) = h(m', {}^{m}n),$

CS 6) $h(m, nn') = h(m, n)^{n'} = h(m^{n}, n'),$

CS 7) $\lambda(h(m, n)) = m^{n}$*,*

CS 8) $\lambda'(h(m, n)) = {}^{m}n,$

CS 9) $h(\lambda l, n) = l^{n},$

CS 10) $h(m, \lambda' l) = {}^{m}l,$

CS 11) $h(m, n)h(m'', n'') = h(m^{n}, {}^{m''}n'')$

for all $r \in R$*,* $m, m_1, m', m'' \in M$*,* $n, n_1, n', n'' \in N$*,* $p, p' \in P$*,* $l \in L$ *with* $tm = tm_1 = sn = sn_1$*,* $tp = sm, tn = sp'$*,* $tm' = sm$*,* $tn = sn'$*,* $tl = sn$*,* $tm = sl$*,* $tn = sm''$*.*

We will denote such a crossed square with $\begin{pmatrix} L & M \\ N & P \end{pmatrix}$. A morphism of crossed squares, $\Phi : \begin{pmatrix} L & M \\ N & P \end{pmatrix} \longrightarrow \begin{pmatrix} L' & M' \\ N' & P' \end{pmatrix}$ consists of four R-algebroid morphisms $\Phi_L : L \longrightarrow L'$, $\Phi_M : M \longrightarrow M'$, $\Phi_N : N \longrightarrow N'$ and $\Phi_P : P \longrightarrow P'$ such that: the resulting cube of R-algebroid morphisms is commutative; $\Phi_L(h(m, n)) = h(\Phi_M(m), \Phi_N(n))$ for $m \in M$, $n \in N$; each of the morphisms Φ_L, Φ_M and Φ_N preserve the action of Φ_P. Thus, all R-algebroid crossed squares and their morphisms form a category denoted by **XSqua**.

3. From Simp.R-Alg.$_{\leq 2}$ to XSqua

In this section, we will obtain a crossed square of R-algebroids from a simplicial R-algebroid $\mathbf{E} = (E_n, d_i^n, s_j^n)$ with Moore complex of lenght 2.

Proposition 3.1. *Given a simplicial R-algebroid* $\boldsymbol{E} = (E_n, d_i^n, s_j^n)$ *with Moore complex of lenght 2, we obtain a crossed square of R-algebroids.*

Proof 3.2. *Let* $\boldsymbol{E} = (E_n, d_i^n, s_j^n)$ *be a simplicial R-algebroid with Moore complex of lenght 2. Then for the simplicial R-algebroid*

$$\boldsymbol{E} \;=\; \cdots\cdots\; E_3 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; E_2 \;\substack{\xrightarrow{d_2}\\ \xrightarrow{d_1}\\ \xrightarrow{d_0}\\ \xleftarrow{s_0}\\ \xleftarrow{s_1}}\; E_1 \;\substack{\xrightarrow{d_1}\\ \xrightarrow{d_0}\\ \xleftarrow{s_0}}\; E_0$$

its Moore complex is as follows,

$$\ldots \to 0 \to 0 \to NE_2 \xrightarrow{d_2^2} NE_1 \xrightarrow{d_1^1} NE_0 = E_0$$

Where $NE_2 = kerd_0^2 \cap kerd_1^2$, $NE_1 = kerd_0^1$ *and* $NE_0 = E_0$. *Let* $\overline{NE_1} = \ker d_1^1$.

We consider the following diagram

$$\begin{array}{ccc} NE_2 & \xrightarrow{d_2^2} & NE_1 \\ {\scriptstyle d_2^2}\downarrow & & \downarrow{\scriptstyle i} \\ \overline{NE_1} & \xrightarrow[i]{} & E_1 \end{array}$$

E_1 *acts on* NE_1 *and* $\overline{NE_1}$ *as;*

$$\begin{array}{ccccccc} E_1 & \times & NE_1 & \longrightarrow & NE_1 \\ (e & , & x) & \mapsto & {}^e x = ex \end{array} \qquad \begin{array}{ccccccc} NE_1 & \times & E_1 & \longrightarrow & NE_1 \\ (x & , & e') & \mapsto & x^{e'} = xe' \end{array}$$

and

$$\begin{array}{ccccccc} E_1 & \times & \overline{NE_1} & \longrightarrow & \overline{NE_1} \\ (e & , & y) & \mapsto & {}^e y = ey \end{array} \qquad \begin{array}{ccccccc} \overline{NE_1} & \times & E_1 & \longrightarrow & \overline{NE_1} \\ (y & , & e') & \mapsto & y^{e'} = ye' \end{array}$$

E_1 *acts on* NE_2

$$\begin{array}{ccccccc} E_1 & \times & NE_2 & \longrightarrow & NE_2 \\ (e & , & a) & \mapsto & {}^e a = s_1^1(e)a \end{array} \qquad \begin{array}{ccccccc} NE_2 & \times & E_1 & \longrightarrow & NE_2 \\ (a & , & e') & \mapsto & a^{e'} = as_1^1(e') \end{array}$$

Also NE_1 *acts on* $\overline{NE_1}$

$$\begin{array}{ccccccc} NE_1 & \times & \overline{NE_1} & \longrightarrow & \overline{NE_1} \\ (x & , & y) & \mapsto & {}^x y = xy \end{array} \qquad \begin{array}{ccccccc} \overline{NE_1} & \times & NE_1 & \longrightarrow & \overline{NE_1} \\ (y & , & x') & \mapsto & y^{x'} = yx' \end{array}$$

and $\overline{NE_1}$ *acts on* NE_1

$$\begin{array}{ccccccc} \overline{NE_1} & \times & NE_1 & \longrightarrow & NE_1 \\ (y & , & x) & \mapsto & {}^y x = yx \end{array} \qquad \begin{array}{ccccccc} NE_1 & \times & \overline{NE_1} & \longrightarrow & NE_1 \\ (x & , & y') & \mapsto & x^{y'} = xy' \end{array}$$

and consider the map

$$\begin{array}{rcccc} h: & NE_1 \times \overline{NE_1} & \longrightarrow & NE_2 \\ & (x,y) & \mapsto & h(x,y) = s_1^1(x)(s_1^1(y) - s_0^1(y)) \end{array}$$

Thus the diagram

$$\begin{array}{ccc} NE_2 & \xrightarrow{d_2^2} & NE_1 \\ {\scriptstyle d_2^2}\downarrow & & \downarrow{\scriptstyle i} \\ \overline{NE_1} & \xrightarrow[i]{} & E_1 \end{array}$$

is a crossed square.

ÇK 1) $d_2^2 : NE_2 \longrightarrow NE_1$ *için*

$$CM\ 1)\ d_2^2({}^{x}a) = d_2^2(s_1^1(x)a) = d_2^2 s_1^1(x) d_2^2(a) = x d_2^2(a)$$

$$\begin{aligned} CM\ 2)\ {}^{d_2^2(a)}a' &= s_1^1 d_2^2(a)a' = d_3^3 s_1^2(a) d_3^3 s_2^2(a') - aa' + aa' \\ &= d_3^3 s_1^2(a) d_3^3 s_2^2(a') - d_3^3 s_2^2(a) d_3^3 s_2^2(a') + aa' \\ &= d_3^3(s_1^2(a)s_2^2(a') - s_2^2(a)s_2^2(a')) + aa' \\ &= aa' \end{aligned}$$

where $s_1^2(a)s_2^2(a') - s_2^2(a)s_2^2(a') \in NE_3 = \{0\}$. *Thus* $d_2^2 : NE_2 \longrightarrow NE_1$ *is a crossed module. Similarly it can be show that the morphisms* $d_2^2 : NE_2 \longrightarrow \overline{NE_1}$, $i : NE_1 \longrightarrow E_1$, $i : \overline{NE_1} \longrightarrow E_1$ *ve* $id_2^2 : NE_2 \longrightarrow E_1$ *are crossed modules.*

ÇK 2)

$$\begin{aligned} h(x+x',y) &= s_1^1(x+x')(s_1^1(y) - s_0^1(y)) \\ &= s_1^1(x)(s_1^1(y) - s_0^1(y)) + s_1^1(x')(s_1^1(y) - s_0^1(y)) \\ &= h(x,y) + h(x',y) \end{aligned}$$

and

$$\begin{aligned} h(x,y+y') &= s_1^1(x)(s_1^1(y+y') - s_0^1(y+y')) \\ &= s_1^1(x)(s_1^1(y) - s_0^1(y)) + s_1^1(x)(s_1^1(y') - s_0^1(y')) \\ &= h(x,y) + h(x,y'). \end{aligned}$$

ÇK 3) *For* $r \in R$,

$$\begin{aligned} r \centerdot h(x,y) &= r \centerdot s_1^1(x)(s_1^1(y) - s_0^1(y)) \\ &= s_1^1(r \centerdot x)(s_1^1(y) - s_0^1(y)) \\ &= h(r \centerdot x, y) \\ &= s_1^1(x)(r \centerdot s_1^1(y) - r \centerdot s_0^1(y)) \\ &= s_1^1(x)(s_1^1(r \centerdot y) - s_0^1(r \centerdot y)) \\ &= h(x, r \centerdot y). \end{aligned}$$

ÇK 4) *For* $e \in E_1$

$$\begin{aligned} {}^{e}h(x,y) &= s_1^1(e)(s_1^1(x)(s_1^1(y) - s_0^1(y))) \\ &= s_1^1(ex)(s_1^1(y) - s_0^1(y))) \\ &= s_1^1({}^{e}x)(s_1^1(y) - s_0^1(y))) \\ &= h({}^{e}x, y), \end{aligned}$$

similarly we get $h(x,y)^{e'} = h(x, y^{e'})$.

ÇK 5)

$$\begin{aligned} h(xx', y) &= s_1^1(x)s_1^1(x')(s_1^1(y) - s_0^1(y)) \\ &= s_1^1(x)(s_1^1(x')s_1^1(y) - s_1^1(x')s_0^1(y)) \\ &= s_1^1(x)(s_1^1(x')s_1^1(y) - s_1^1(x')s_0^1(y) + s_0^1(x')s_0^1(y) - \\ &\quad s_0^1(x')s_0^1(y)) \\ &= s_1^1(x)(s_1^1(x'y) - s_0^1(x'y)) - s_1^1(x)(s_1^1(x') - s_0^1(x'))(s_0^1(y) - \\ &\quad s_1^1s_0^0d_1^1(y)) \\ &= h(x,\ {}^{x'}y) - d_3^3[s_2^2s_1^1(x)(s_2^2s_1^1(x') - s_2^2s_0^1(x'))(s_2^2s_0^1(y) - \\ &\quad s_1^2s_0^1(y))] \\ &= h(x,\ {}^{x'}y) \end{aligned}$$

where $s_2^2s_1^1(x)(s_2^2s_1^1(x') - s_2^2s_0^1(x'))(s_2^2s_0^1(y) - s_1^2s_0^1(y)) \in NE_3 = \{0\}$.

ÇK 6) *Similarly we can show that*

$$h(x, yy') = h(x^y, y').$$

ÇK 7- ÇK 8)

$$\begin{aligned} d_2^2h(x,y) &= d_2^2(s_1^1(x)(s_1^1(y) - s_0^1(y))) \\ &= d_2^2s_1^1(x)(d_2^2s_1^1(y) - d_2^2s_0^1(y))) \\ &= x(y - s_0^0d_1^1(y)) \quad \textit{(}d_1^1(y) = 0\textit{)} \\ &= xy = x^y \\ &= {}^{x}y \end{aligned}$$

ÇK 9 - ÇK 10)

$$\begin{aligned} h(d_2^2(a), y) &= s_1^1(d_2^2(a))(s_1^1(y) - s_0^1(y)) \\ &= s_1^1d_2^2(a)(s_1^1(y) - s_0^1(y)) - as_1^1(y) + as_1^1(y) + \\ &\quad as_1^1s_0^0d_1^1(y), \quad \textit{(}d_1^1(y) = 0\textit{)} \\ &= d_3^3s_1^2(a)(d_3^3s_2^2s_1^1(y) - d_3^3s_2^2s_0^1(y)) - d_3^3s_2^2(a)d_3^3s_2^2s_1^1(y) + \\ &\quad d_3^3s_2^2(a)d_3^3s_1^2s_0^1(y) + as_1^1(y) \\ &= d_3^3(s_1^2(a)s_2^2s_1^1(y) - s_2^2s_0^1(y)) - s_2^2(a)s_2^2s_1^1(y) + \\ &\quad s_2^2(a)s_1^2s_0^1(y)) + a^y \\ &= a^y \end{aligned}$$

where $s_1^2(a)s_2^2s_1^1(y) - s_2^2s_0^1(y)) - s_2^2(a)s_2^2s_1^1(y) + s_2^2(a)s_1^2s_0^1(y) \in NE_3 = \{0\}$ *and*

$$\begin{aligned} h(x, d_2^2(a)) &= s_1^1(x)(s_1^1(d_2^2(a)) - s_0^1(d_2^2(a))) \\ &= s_1^1(x)(s_1^1 d_2^2(a) - s_0^1 d_2^2(a) - a + a) \\ &= d_3^3 s_2^2 s_1^1(x)(d_3^3 s_1^2(a) - d_3^3 s_0^2(a) - d_3^3 s_1^2(a)) + s_1^1(x)a \\ &= d_3^3(s_2^2 s_1^1(x)(s_1^2(a) - s_0^2(a) - s_1^2(a)) + {}^{x}a \\ &= {}^{x}a \end{aligned}$$

where $s_2^2s_1^1(x)(s_1^2(a) - s_0^2(a) - s_1^2(a) \in NE_3 = \{0\}$.

ÇK 11)

$$\begin{aligned} h(x,y)h(x',y') &= s_1^1(x)(s_1^1(y) - s_0^1(y))s_1^1(x')(s_1^1(y') - s_0^1(y')) \\ &= [s_1^1(x)s_1^1(y) - s_1^1(x)s_0^1(y)][s_1^1(x')s_1^1(y') - s_1^1(x')s_0^1(y')] \\ &= s_1^1(x)s_1^1(y)s_1^1(x')s_1^1(y') - s_1^1(x)s_1^1(y)s_1^1(x')s_0^1(y') - \\ &\quad s_1^1(x)s_0^1(y)s_1^1(x')s_1^1(y') + s_1^1(x)s_0^1(y)s_1^1(x')s_0^1(y') \\ &= s_1^1(x)s_1^1(y)s_1^1(x')s_1^1(y') - s_1^1(x)s_1^1(y)s_1^1(x')s_0^1(y') - \\ &\quad s_1^1(x)s_0^1(y)s_1^1(x')s_1^1(y') + s_1^1(x)s_0^1(y)s_1^1(x')s_0^1(y') + \\ &\quad s_1^1(x)s_1^1(y)s_0^1(x')s_0^1(y') - s_1^1(x)s_1^1(y)s_0^1(x')s_0^1(y') \\ &= s_1^1(x)s_1^1(y)s_1^1(x')s_1^1(y') - s_1^1(x)s_1^1(y)s_0^1(x')s_0^1(y') \\ &\quad -s_1^1(x)s_1^1(y)s_1^1(x')s_0^1(y') - s_1^1(x)s_0^1(y)s_1^1(x')s_1^1(y') + \\ &\quad s_1^1(x)s_0^1(y)s_1^1(x')s_0^1(y') + s_1^1(x)s_1^1(y)s_0^1(x')s_0^1(y') \\ &= s_1^1(xy)(s_1^1(x'y') - s_0^1(x'y')) \\ &= h(xy, x'y') = h(x^y, {}^{x'}y') \end{aligned}$$

where

$$\begin{aligned} &-s_1^1(x)s_1^1(y)s_1^1(x')s_0^1(y') - s_1^1(x)s_0^1(y)s_1^1(x')s_1^1(y') + s_1^1(x)s_0^1(y)s_1^1(x')s_0^1(y') + \\ &s_1^1(x)s_1^1(y)s_0^1(x')s_0^1(y') \in d_3^3(NE_3). \end{aligned}$$

Thus we have a crossed square.

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