NEW APPROACHES IN NATURAL SCIENCES AND MATHEMATICS:

THEORY, METHOD, AND PRACTICE

Editor: Assoc. Prof. Dr. Füsun Şeyma KIŞKAN

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853 Sokak No:13 P.10 Kemeraltı-Konak/İzmir
Tel: 0 232 484 88 68
www.duvaryayinlari.com
duvarkitabevi@gmail.com

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Chapter 1

On Partner Ruled Surfaces Generated by the T and Nq Quasi-Vectors

Başak YAĞBASAN[1], Cumali EKİCİ[2]

ABSTRACT

In this study, the definitions and theorems related to the quasi-frame of a space curve in Euclidean 3-space are presented. For a curve equipped with a quasi-frame, the quasi-vectors, the quasi-derivative equations, and the quasi-curvatures k_1, k_2 and k_3 are introduced. Using these tools, the partner ruled surfaces generated by the quasi-vectors $\boldsymbol{T}(t)$ and $\boldsymbol{N}_q(t)$ of directed curves in Euclidean space, referred to as directional partner ruled surfaces are investigated. Furthermore, the first and second fundamental forms of these surfaces, as well as their Gaussian and mean curvatures, the distribution parameter, and the striction line are derived.

Keywords: Euclidean 3-space, quasi-frame, partner ruled surface, curvatures.

T ve Nq QUASİ-VEKTÖRLERİ TARAFINDAN ÜRETİLEN ORTAK REGLE YÜZEYLER ÜZERİNE

ÖZET

Bu çalışmada Öklidiyen 3-uzaydaki bir uzay eğrisi için quasi-çatı için tanımlar ve teoremler verilmiştir. Quasi-çatılı bir eğri için quasi-vektörleri, quasi-türev denklemleri ve eğrinin k_1, k_2 ve k_3 quasi-eğriliklerinden bahsedilmiştir. Bunlar yardımıyla Öklid uzayında yönlü ortak regle yüzeyler olarak adlandırılan Öklid uzayındaki yönlü eğriler için hesaplanan $\boldsymbol{T}(t)$ and $\boldsymbol{N}_q(t)$ quasi vektörleri ile oluşturulan ortak regle yüzeyler incelenmiştir. Ayrıca, bu yönlü ortak regle yüzeylerin birinci, ikinci temel formları, Gauss ve ortalama eğrilikleri ile bu regle yüzeylerin dağılma parametresi ve striksiyon çizgisi elde edilmiştir.

Anahtar Kelimeler: Öklidiyen 3-uzay, quasi-çatı, partner regle yüzey, eğrilikler

[1] Eskişehir Osmangazi University, Department of Mathematics and Computer Science, Eskişehir. bskyagbasan@gmail.com, 05315724143
ORCID: 0000-0003- 4067-3034

[2] Eskişehir Osmangazi University, Department of Mathematics and Computer Science, Eskişehir. cekici@ogu.edu.tr, 05323056666
ORCID: 0000-0002-3247-5727

INTRODUCTION

Ruled surfaces represent a fundamental class of objects in both classical and modern differential geometry, owing to their structural simplicity and extensive geometric applications. The geometry and characterization of ruled surfaces within Euclidean, Galilean, and Minkowski spaces have been thoroughly examined in the existing literature (Gray, 1993; Izumiya and Takeuchi, 2003; Yu et al., 2014; Dede and Ekici, 2016; Ekici et al., 2020). Hussein and Youssef 2016, Soliman 2018, Yoon et al. 2019 and Kaymanlı et al. 2020 studied the evolutions of the ruled surfaces via the evolution of their directrix. Numerous specialized classes, including developable ruled surfaces, offset surfaces, canal and tubular surfaces, and directional or quasi-generated surfaces, have been introduced to enhance our understanding of their intrinsic and extrinsic characteristics (Coquillart, 1987; Alegre et al., 2010; Dede and Ekici, 2011, 2016). Dede et al. found quasi frame in 2015 and Soliman also used this frame to work on this subject in 2018 (Dede et al., 2015; Soliman et al., 2018).

The ruled surfaces produced by alternate moving frames, such as the Darboux, Bishop, quasi-frame, and Flc frames, have received special attention in recent years. These frames offer fresh insights into the curvature structures and developability conditions of families of ruled surfaces and enable simultaneous or partner characterizations (Ünlütürk et al., 2016; Şentürk and Yüce, 2015; Ouarab et al., 2018; Masal and Azak, 2018; Ouarab, 2021; Kaymanlı et al., 2022; Li et al., 2022, 2023). Additionally, new classifications of parallel, directional, and generalized ruled surfaces have been produced using the directional and quasi-vector techniques presented in Euclidean and Galilean spaces (Ekici et al., 2020, 2021; Dede et al., 2024).

Further research into the reconstruction, characterisation, and simultaneous evolution of ruled surfaces with regard to different frame structures has been spurred by these advancements. Specifically, ruled surfaces and tube surfaces, and their geometric invariants in three- and four-dimensional ambient spaces may be examined using a unifying mechanism that makes use of quasi-vectors and quasi-frame components (Soliman et al., 2018; Kaymanlı et al., 2020, 2022; Ekici et al., 2021; Yağbasan et al., 2023; Ekici Coşkun and Akça, 2023).

The study of partner ruled surfaces, constructed from polynomial curves, has been a recent subject of geometric analysis (Ouarab, 2021; Li et al., 2022). Notably, Li et al. utilized the Flc frame to characterize these surfaces (Li et al., 2022). Simultaneously, Soukaina's work focused on determining the surfaces' developability properties via the Darboux frame (Ouarab, 2021). Cengiz's work involved the investigation of partner ruled surfaces, specifically utilizing the characteristics of the Euclidean hybrid frame and the Lorentz hybrid frame

(Cengiz, 2025). By investigating novel geometric characteristics of ruled surfaces produced by quasi-frame vectors and associated partner constructions, the current study seeks to add to this expanding corpus of work.

PRELIMINARIES

Let $\boldsymbol{X} = (x_1, x_2, x_3)$ and $\boldsymbol{Y} = (y_1, y_2, y_3)$ be two vectors in $\mathbb{E}^3$. The dot product is defined herein using the notation $< \boldsymbol{X}, \boldsymbol{Y} >= x_1y_1 + x_2y_2 + x_3y_3$, the Euclidean norm of a vector is expressed as $\|\boldsymbol{X}\| = \sqrt{< \boldsymbol{X}, \boldsymbol{X} >}$, and the cross product is

$$\boldsymbol{X} \wedge \boldsymbol{Y} = (x_2y_3 - x_3y_2)\boldsymbol{e}_1 - (x_3y_1 - x_1y_3)\boldsymbol{e}_2 + (x_1y_2 - x_2y_1)\boldsymbol{e}_3$$

predicated on $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ being an orthonormal basis (Do Carmo, 1976; Gray, 1993). In place of the classical Frenet-Serret apparatus, an alternative method utilizes a new adapted basis, termed the quasi-frame, which is defined along a rectifiable spatial curve $\alpha(t)$. The set of orthonormal vectors constituting this frame consists of the unit tangent vector $\boldsymbol{T}$, the quasi-normal vector $\mathbf{N}_q$, and the third fundamental direction, the quasi-binormal vector $\mathbf{B}_q$. The quasi-frame $\{\boldsymbol{T}, \mathbf{N}_q, \boldsymbol{B}_q, \mathbf{k}\}$ is given by

$$\boldsymbol{T} = \frac{\alpha'}{||\alpha'||}, \qquad \mathbf{N}_q = \frac{\boldsymbol{T} \wedge \mathbf{k}}{|| \boldsymbol{T} \wedge \mathbf{k}||}, \quad \mathbf{B}_q = \boldsymbol{T} \wedge \mathbf{N}_q \tag{01}$$

where $\mathbf{k}$ is the projection vector. We have initially specified the projection direction $\mathbf{k}$ to be $(1,0,0)$ for computational convenience. However, a singularity in the quasi-frame construction emerges precisely when $\boldsymbol{T}$ and $\mathbf{k}$ are aligned. Therefore, in those specific circumstances where the unit tangent $\boldsymbol{T}$ is parallel to $\mathbf{k}$, an alternate projection vector, namely $\mathbf{k} = (0,1,0)$, is adopted (Dede et al.,2015). The equations of variation for the spatial curve (or spatial spring curve) parametrized by the spring variable are derived as

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}_q' \\ \boldsymbol{B}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N}_q \\ \boldsymbol{B}_q \end{bmatrix} \tag{02}$$

where the functions

$$k_1 = \langle \boldsymbol{T}', \boldsymbol{N}_q \rangle, \qquad k_2 = \langle \boldsymbol{T}', \boldsymbol{B}_q \rangle, \qquad k_3 = \langle \boldsymbol{N}_q', \boldsymbol{B}_q \rangle \tag{03}$$

respectively (Dede et al.,2015; Kaymanlı et al., 2020; Ekici et al., 2021; Ekici and Akça, 2023).

The parametric equation of ruled surface $\phi(t, v)$ is given as

$$\phi(t, v) = \alpha(t) + v.\mathbf{X}(t) \tag{04}$$

where $\alpha(t)$ is a curve and $\mathbf{X}(t)$ is a generator vector (Do Carmo, 1976; Gray, 1993; Yu et al., 2014). The distribution parameter of the ruled surface is identified by

$$P_{\boldsymbol{X}} = \frac{\det(\alpha_t, \boldsymbol{X}, \boldsymbol{X}_t)}{\langle \boldsymbol{X}_t, \boldsymbol{X}_t \rangle}. \tag{05}$$

The central point (or striction point) on a ruled surface corresponds to the projection onto the generating line of the foot of the segment which establishes the shortest distance between two consecutive generators. It is given as

$$\beta_{\boldsymbol{X}}(t) = \alpha(t) - \frac{\langle \alpha_t, \boldsymbol{X}_t \rangle}{\langle \boldsymbol{X}_t, \boldsymbol{X}_t \rangle} \boldsymbol{X}(t). \tag{06}$$

(Do Carmo, 1976; Gray, 1993; Dede et al., 2024).

Let $\mathcal{M}$ denote a differentiable surface embedded in $\mathbb{R}^3$, which is locally defined by the parameterization $\phi: U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\phi(t, v)$. The tangent space to $\mathcal{M}$ at an arbitrary point $p = \phi(t, v)$ in $\mathcal{M}$ is spanned by the vector space $\{\phi_t, \phi_v\}$. The coefficients of the first fundamental form of $\mathcal{M}$ are defined as

$$E = \langle \phi_t, \phi_t \rangle, F = \langle \phi_t, \phi_v \rangle, G = \langle \phi_v, \phi_v \rangle \text{ and } W = EG - F^2 \tag{07}$$

where $<,>$ is the Euclidean inner product (Do Carmo, 1976; Gray, 1993). Then the unit normal vector field of $\mathcal{M}$ is defined as

$$\boldsymbol{U} = \frac{\phi_t \wedge \phi_v}{||\phi_t \wedge \phi_v||}.$$

The coefficients of its second fundamental form of $\mathcal{M}$ are defined as

$$L = \langle \phi_{tt}, \boldsymbol{U} \rangle, \qquad M = \langle \phi_{tv}, \boldsymbol{U} \rangle \text{ and } N = \langle \phi_{vv}, \boldsymbol{U} \rangle. \tag{08}$$

By convention, the intrinsic curvature invariant (Gaussian curvature) and the extrinsic curvature invariant (mean curvature) of the surface are assigned the notations

$$K = \frac{LN - M^2}{EG - F^2} \text{ and } H = \frac{LG - 2MF + NE}{EG - F^2}, \tag{09}$$

in that order (Do Carmo, 1976; Gray, 1993; Kaymanlı et al., 2022).

ON PARTNER RULED SURFACES GENERATED BY THE T AND Nq QUASI-VECTORS

Let $\alpha: t \in I \to \alpha(t)$ be a C^2-class differentiable unit speed curve lying on a ruled surface $\phi(t, v)$. Let denote by $\{\boldsymbol{T}(t), \boldsymbol{N}_q(t), \boldsymbol{B}_q(t)\}$ the quasi-frame of $\alpha = \alpha(t)$ on $\phi(t, v)$. The two partner ruled surfaces defined by

$$^{\boldsymbol{TN}_q}\phi(t, v) = \boldsymbol{T}(t) + v\boldsymbol{N}_q(t) \tag{10}$$

and

$$^{\boldsymbol{N}_q\boldsymbol{T}}\phi(t, v) = \boldsymbol{N}_q(t) + v\boldsymbol{T}(t) \tag{11}$$

are called $\boldsymbol{TN}_q$ or $\boldsymbol{N}_q\boldsymbol{T}$ partner ruled surfaces according to the quasi-frame of the curve $\alpha(t)$ on the surface $\phi(t, v)$.

Partner ruled surfaces with $\boldsymbol{T}$ and $\boldsymbol{N}_q$ quasi-vectors in 3-dimensional space

The essential theorems and their corresponding proofs pertaining to the parametric equation are provided here in.

Theorem 1 Let $\mathcal{M} \subset \mathbb{E}^3$ be a partner ruled surface to quasi-frame $\{\boldsymbol{T}(t), \boldsymbol{N}_q(t), \boldsymbol{B}_q(t)\}$ with parametrization ${}^{\boldsymbol{TN}_q}\phi(t,v)$. The unit normal vector field $\boldsymbol{U}_1(t,v)$ of the partner ruled surface in $\mathbb{E}^3$ is found to be

$$\boldsymbol{U}_1(t,v) = \frac{-(k_2 + vk_3)\boldsymbol{T}(t) - vk_1\boldsymbol{B}_q(t)}{\sqrt{(k_2 + vk_3)^2 + v^2k_1^2}}$$

Proof A partner ruled surface, quasi-frame $\{\boldsymbol{T}(t), \boldsymbol{N}_q(t), \boldsymbol{B}_q(t)\}$ is parametrized by

$${}^{\boldsymbol{TN}_q}\phi(t,v) = \boldsymbol{T}(t) + r\boldsymbol{N}_q(t).$$

The partial derivatives of ${}^{\boldsymbol{TN}_q}\phi(t,v)$, with respect to t and v, are determined by

$${}^{\boldsymbol{TN}_q}\phi_t(t,v) = -vk_1\boldsymbol{T}(t) + k_1\boldsymbol{N}_q(t) + (k_2 + vk_3)\boldsymbol{B}_q(t) \tag{12}$$

and

$${}^{\boldsymbol{TN}_q}\phi_v(t,v) = \boldsymbol{N}_q(t). \tag{13}$$

Then, second order partial derivatives of ${}^{\boldsymbol{TN}_q}\phi(t,v)$, with respect to t and v, are given as

$$\begin{aligned} {}^{\boldsymbol{TN}_q}\phi_{tt}(t,v) = &\quad -(k_1^2 + k_2^2 + vk_1' + vk_2k_3)\boldsymbol{T}(t) + (k_1' - k_2k_3 - vk_1^2 \\ &\quad -vk_2^2)\boldsymbol{N}_q(t) + (k_1k_3 + k_2' - vk_1k_2 + vk_3')\boldsymbol{B}_q(t) \\ {}^{\boldsymbol{TN}_q}\phi_{tv}(t,v) = &\quad -k_1\boldsymbol{T}(t) + k_3\boldsymbol{B}_q(t) \end{aligned} \tag{14}$$

and

$${}^{\boldsymbol{TN}_q}\phi_{vv}(t,v) = 0. \tag{15}$$

The unit normal vector field $\boldsymbol{U}_1(t,v)$ of this surface should be provided with the following conditions

$$\begin{gathered} < {}^{\boldsymbol{TN}_q}\phi_t(t,v), \boldsymbol{U}_1(t,v) >= 0, \\ < {}^{\boldsymbol{TN}_q}\phi_v(t,v), \boldsymbol{U}_1(t,v) >= 0, \\ < \boldsymbol{U}_1(t,v), \boldsymbol{U}_1(t,v) >= 1 \end{gathered} \tag{16}$$

where ${}^{\boldsymbol{TN}_q}\phi_t(t,v)$ and ${}^{\boldsymbol{TN}_q}\phi_v(t,v)$ are the partial derivatives of ${}^{\boldsymbol{TN}_q}\phi(t,v)$. The unit normal vector field $\boldsymbol{U}_1(t,v)$ of partner ruled surface is obtained as

$$\boldsymbol{U}_1(t,v) = \frac{-(k_2 + vk_3)\boldsymbol{T}(t) - vk_1\boldsymbol{B}_q(t)}{\sqrt{(k_2 + vk_3)^2 + v^2k_1^2}} \tag{17}$$

Theorem 2 Let $\mathcal{M}$ be a partner ruled surface in $\mathbb{E}^3$ associated with the quasi-frame $\{\boldsymbol{T}(t), \boldsymbol{N}_q(t), \boldsymbol{B}_q(t)\}$, parameterized by ${}^{\boldsymbol{TN}_q}\phi(t,v)$. Gaussian curvature K_1 and mean curvature H_1 of the partner ruled surface with unit normal vector field are obtained as

$$K_1 = \frac{-k_1^2 k_2^2}{((k_2 + vk_3)^2 + v^2 k_1^2)^2}$$

and

$$H_1 = \frac{-k_1^2 k_2 + k_2^3 + vk_2^2(2k_3 + (\frac{k_1}{k_2})') + v^2((\frac{k_1}{k_3})' k_3^2 + k_2(k_1^2 + k_3^2))}{((k_2 + vk_3)^2 + v^2 k_1^2)^{3/2}},$$

respectively.

Proof Using equation (16), by substituting (12) and (13) into equation (07), the coefficients of the first fundamental form for the partner ruled surface

$$\begin{aligned} E_1 &= v^2 k_1^2 + k_1^2 + k_2^2 + 2vk_2k_3 + v^2 k_3^2 \\ F_1 &= k_1 \\ G_1 &= 1 \end{aligned} \tag{18}$$

and

$$W_1 = v^2 k_1^2 + k_2^2 + 2vk_2k_3 + v^2 k_3^2$$

are subsequently derived. Equations (08), (14), (15) and (17) lead to the coefficients of the second fundamental form of the partner ruled surface with the unit vector field in $\mathbb{E}^3$ obtained as,

$$\begin{aligned} L_1 &= \frac{(k_1^2 + k_2^2 + vk_1' + vk_2k_3)(k_2 + vk_3) + vk_1(k_1k_3 + k_2' - vk_1k_2 + vk_3')}{\sqrt{(k_2 + vk_3)^2 + v^2 k_1^2}} \\ M_1 &= \frac{k_1 k_2}{\sqrt{(k_2 + vk_3)^2 + v^2 k_1^2}} \\ N_1 &= 0 \end{aligned} \tag{19}$$

Substituting equations (18) and (19) into equation (09) implies that Gaussian and mean curvatures with respect to $\boldsymbol{U}_1(t, v)$ following as

$$K_1 = \frac{-k_1^2 k_2^2}{((k_2 + vk_3)^2 + v^2 k_1^2)^2}$$

and

$$H_1 = \frac{-k_1^2 k_2 + k_2^3 + vk_2^2(2k_3 + (\frac{k_1}{k_2})') + v^2((\frac{k_1}{k_3})' k_3^2 + k_2(k_1^2 + k_3^2))}{((k_2 + vk_3)^2 + v^2 k_1^2)^{3/2}}.$$

Theorem 3 The striction curves and distribution parameter on the partner ruled surface using the $\boldsymbol{T}'(t), \boldsymbol{B}_q(t)$ and $\boldsymbol{B}_q'(t)$ are given by

$$\beta_{\boldsymbol{TN}_q}(t) = \boldsymbol{T}(t) - \frac{k_2 k_3}{k_1^2 + k_3^2} \boldsymbol{N}_q(t)$$

and

$$P_{\boldsymbol{TN}_q} = \frac{k_1 k_2}{k_1^2 + k_3^2}$$

respectively.

Proof Considering equation (06), the striction curve can be expressed as

$$\beta_{\boldsymbol{TN}_q}(t) = \boldsymbol{T}(t) - \frac{\langle \boldsymbol{T}'(t), \boldsymbol{N}_q'(t)\rangle}{\langle \boldsymbol{N}_q'(t), \boldsymbol{N}_q'(t)\rangle}\boldsymbol{N}_q(t).$$

By applying equation (02), the desired result is obtained. Similarly, considering equation (05), the distribution parameter is given by

$$P_{\boldsymbol{TN}_q} = \frac{\det\left(\boldsymbol{T}(t), \boldsymbol{N}_q(t), \boldsymbol{N}_q'(t)\right)}{\langle \boldsymbol{N}_q'(t), \boldsymbol{N}_q'(t)\rangle}.$$

Using equation (02) again, the desired result is readily obtained.

Partner ruled surfaces with $\boldsymbol{N_q}$ and $\boldsymbol{T}$ quasi-vectors in 3-dimensional space

The partner ruled surface defined by

$$^{\boldsymbol{N}_q\boldsymbol{T}}\phi(t,v) = \boldsymbol{N}_q(t) + v\boldsymbol{T}(t) \tag{20}$$

are called $\boldsymbol{N}_q\boldsymbol{T}$ partner ruled surfaces according to the quasi-frame of the curve $\alpha(t)$ on the surface $\phi(t,v)$.

Theorem 4 Let $\mathcal{M}$ be a partner ruled surface in 3-dimensional space associated with the quasi-frame $\{\boldsymbol{T}(t), \boldsymbol{N}_q(t), \boldsymbol{B}_q(t)\}$, parameterized by $^{\boldsymbol{N}_q\boldsymbol{T}}\phi(t,v)$. Unit normal vector field $\boldsymbol{U}_2(t,v)$ of the partner ruled surface in $\mathbb{E}^3$ is obtained as

$$\boldsymbol{U}_2(t,v) = \frac{(k_3 + vk_2)\boldsymbol{N}_q(t) - vk_1\boldsymbol{B}_q(t)}{\sqrt{(k_3 + vk_2)^2 + v^2k_1^2}}$$

Proof A partner ruled surface with quasi-frame $\{\boldsymbol{T}(t), \boldsymbol{N}_q(t), \boldsymbol{B}_q(t)\}$ are parametrized by

$$^{\boldsymbol{N}_q\boldsymbol{T}}\phi(t,v) = \boldsymbol{N}_q(t) + v\boldsymbol{T}(t).$$

The partial derivatives of $^{\boldsymbol{N}_q\boldsymbol{T}}\phi(t,v)$, with respect to t and v, are determined by

$$^{\boldsymbol{N}_q\boldsymbol{T}}\phi_t(t,v) = -k_1\boldsymbol{T}(t) + vk_1\boldsymbol{N}_q(t) + (k_3 + vk_2)\boldsymbol{B}_q(t) \tag{21}$$

and

$$^{\boldsymbol{N}_q\boldsymbol{T}}\phi_v(t,v) = \boldsymbol{T}(t). \tag{22}$$

Then, second order partial derivatives of $^{\boldsymbol{N}_q\boldsymbol{T}}\phi(t,v)$, with respect to t and v, are given as

$$
\begin{aligned}
{}^{TN_q}\phi_{tt}(t,v) = \quad & -(k_1' + vk_1^2 + vk_2^2 + k_2k_3)\boldsymbol{T}(t) - (k_1^2 + k_3^2 - vk_1' \\
& +vk_2k_3)\boldsymbol{N}_q(t) + (k_3' - k_1k_2 + +vk_1k_3 - vk_2')\boldsymbol{B}_q(t) \qquad (23) \\
{}^{TN_q}\phi_{tv}(t,v) = \quad & k_1\boldsymbol{N}_q(t) + k_2\boldsymbol{B}_q(t)
\end{aligned}
$$

and

$$
{}^{TN_q}\phi_{vv}(t,v) = 0. \qquad (24)
$$

The unit normal vector field $\boldsymbol{U}_2(t,v)$ of the surface should be provided with the following conditions

$$
\begin{aligned}
&< {}^{N_qT}\phi_t(t,v), \boldsymbol{U}_2(t,v) >= 0, \\
&< {}^{N_qT}\phi_v(t,v), \boldsymbol{U}_2(t,v) >= 0, \qquad (25) \\
&< \boldsymbol{U}_2(t,v), \boldsymbol{U}_2(t,v) >= 1
\end{aligned}
$$

where ${}^{N_qT}\phi_t(t,v)$ and ${}^{N_qT}\phi_v(t,v)$ are the partial derivatives of ${}^{N_qT}\phi(t,v)$. The unit normal vector field $\boldsymbol{U}_2(t,v)$ of the partner ruled surface is obtained as

$$
\boldsymbol{U}_2(t,v) = \frac{(k_3 + vk_2)\boldsymbol{N}_q(t) - vk_1\boldsymbol{B}_q(t)}{\sqrt{(k_3 + vk_2)^2 + v^2k_1^2}}. \qquad (26)
$$

Theorem 5 Let $\mathcal{M}$ be a partner ruled surface in 3-dimensional space associated with the quasi-frame $\{\boldsymbol{T}(t), \boldsymbol{N}_q(t), \boldsymbol{B}_q(t)\}$, parameterized by ${}^{N_qT}\phi(t,v)$. Gaussian curvature K_2 and mean curvature H_2 of the partner ruled surface with unit normal vector field $\boldsymbol{U}_2(t,v)$ in $\mathbb{E}^3$ are obtained as

$$
K_2 = \frac{-k_1^2k_3^2}{((k_3 + vk_2)^2 + v^2k_1^2)^2}
$$

and

$$
H_2 = \frac{-k_1^2k_3 + k_3^3 + vk_3^2(2k_2 - (\frac{k_1}{k_3})') + v^2((\frac{k_1}{k_2})'k_2^2 + k_3(k_1^2 + k_2^2))}{((k_3 + vk_2)^2 + v^2k_1^2)^{3/2}},
$$

respectively.

Proof Using equation (25), by substituting (21) and (22) into equation (07), the coefficients of the first fundamental form for the partner ruled surfaces

$$
\begin{aligned}
E_2 &= v^2k_1^2 + k_1^2 + k_3^2 + 2vk_2k_3 + v^2k_2^2 \\
F_2 &= -k_1 \qquad (27) \\
G_2 &= 1
\end{aligned}
$$

and

$$
W_2 = v^2k_1^2 + k_3^2 + 2vk_2k_3 + v^2k_2^2
$$

are subsequently derived. Equations (08), (23), (24) and (26) lead to the coefficients of the second fundamental form of this surface with the unit vector field $\boldsymbol{U}_2(t,v)$ in $\mathbb{E}^3$ obtained as

$$L_2 = \frac{-(k_1' + vk_1^2 + vk_2^2 + k_2k_3)(k_3 + vk_2) - vk_1(k_3' - k_1k_2 + vk_1k_3 -}{\sqrt{(k_3 + vk_2)^2 + v^2k_1^2}}$$
$$M_2 = \frac{k_1k_3}{\sqrt{(k_3 + vk_2)^2 + v^2k_1^2}} \quad (28)$$
$$N_2 = 0.$$

Substituting equations (27) and (28) into equation (09) implies that Gaussian and mean curvatures with respect to $\boldsymbol{U}_2(t,v)$ following as

$$K_2 = \frac{-k_1^2k_3^2}{((k_3 + vk_2)^2 + v^2k_1^2)^2}$$

and

$$H_2 = \frac{-k_1^2k_3 + k_3^3 + vk_3^2(2k_2 - (\frac{k_1}{k_3})') + v^2((\frac{k_1}{k_2})'k_2^2 + k_3(k_1^2 + k_2^2))}{((k_3 + vk_2)^2 + v^2k_1^2)^{3/2}}.$$

Theorem 6 The striction curves and distribution parameter on the partner ruled surface using the $\boldsymbol{T}'(t), \boldsymbol{B}_q(t)$ and $\boldsymbol{B}_q'(t)$ are given by

$$\beta_{\boldsymbol{N}_q\boldsymbol{T}}(t) = \boldsymbol{N}_q(t) - \frac{k_2k_3}{k_1^2 + k_2^2}\boldsymbol{T}(t)$$

and

$$P_{\boldsymbol{N}_q\boldsymbol{T}} = \frac{k_1k_3}{k_1^2 + k_2^2}$$

respectively.

Proof By considering equation (06), the striction curve is represented as

$$\beta_{\boldsymbol{N}_q\boldsymbol{T}}(t) = \boldsymbol{N}_q(t) - \frac{\langle \boldsymbol{N}_q'(t), \boldsymbol{T}'(t)\rangle}{\langle \boldsymbol{T}'(t), \boldsymbol{T}'(t)\rangle}\boldsymbol{T}(t).$$

Applying equation (02) subsequently yields the desired result. In a similar manner, equation (05) provides the distribution parameter

$$P_{\boldsymbol{N}_q\boldsymbol{T}} = \frac{\det(\boldsymbol{N}_q(t), \boldsymbol{T}(t), \boldsymbol{T}'(t))}{\langle \boldsymbol{T}'(t), \boldsymbol{T}'(t)\rangle},$$

and the application of equation (02) once again facilitates the straight forward derivation of the intended outcome.

Example 1 Let us consider a curve parameterized as

$$\alpha(t) = \left(\frac{4}{5}\cos(t), 1 - \sin(t), -\frac{3}{5}\cos(t)\right). \quad (29)$$

Then, the quasi-vectors of $\alpha(t)$ are given by

$$\boldsymbol{T} = (-\frac{4}{5}\sin(t), -\cos(t), \frac{3}{5}\sin(t))$$
$$\boldsymbol{N}_q = (0, \frac{3\sin(t)}{\sqrt{16\cos^2(t)+9}}, \frac{5\cos(t)}{\sqrt{16\cos^2(t)+9}}) \tag{30}$$
$$\boldsymbol{B}_q = (-\frac{1}{5}\sqrt{16\cos^2(t)+9}, \frac{4\sin(t)\cos(t)}{\sqrt{16\cos^2(t)+9}}, -\frac{12}{5}\frac{\sin^2(t)}{\sqrt{16\cos^2(t)+9}})$$

and from equation (03), quasi-curvatures are given as

$$k_1 = \frac{3}{\sqrt{16\cos^2(t)+9}},$$
$$k_2 = -\frac{4\cos(t)}{\sqrt{16\cos^2(t)+9}} \text{ and } k_3 = \frac{12\sin(t)}{\sqrt{16\cos^2(t)+9}}.$$

Thus, the partner ruled surface generated by the quasi-vectors is given by the parametric form

$$^{TN_q}\phi(t,v) = \left(-\frac{4}{5}\sin(t), -\cos(t) + \frac{3v\sin(t)}{\sqrt{16\cos^2(t)+9}}, \frac{3}{5}\sin(t) + \frac{5v\cos(t)}{\sqrt{16\cos^2(t)+9}}\right).$$

The unit normal vector field of the $^{TN_q}\phi(t,v)$ surface is thus expressed as

$$\boldsymbol{U}_1 = \frac{\sqrt{16\cos^2(t)+9}}{\Delta}\left(\frac{16\sin(t)\cos(t)\sqrt{16\cos^2(t)+9}+75v}{5\sqrt{16\cos^2(t)+9}}, 4\cos^2(t), -\frac{12\sin(t)\cos(t)}{5}\right)$$

in equation (17), where $\Delta^2 = 96v\sin(t)\cos(t)\sqrt{16\cos^2(t)+9} + 225v^2 + 144\cos^2(t) + 256\cos^4(t)$.

Equations (09) of the partner ruled surface yield the Gaussian and mean curvatures as

$$K_1 = -\frac{144}{\Delta^4}\cos^2(t)\,(16\cos^2(t)+9)^2$$

$$H_1 = \frac{4}{\Delta^3}((256\cos^5(t) - 81\cos(t))\sqrt{16\cos^2(t)+9} + 2304(\cos^2(t)+\frac{3}{16})(\cos^2(t)+\frac{9}{16})v\sin(t))$$

respectively.

The striction curves and distribution parameter on the partner ruled surface $^{TN_q}\phi(t,v)$ are given by

$$\beta_{TN_q}(t) = \left(-\frac{4}{5}\sin(t), -\frac{16}{25}cos^3(t) - \frac{9}{25}\cos(t), \frac{3}{5}\sin(t) + \frac{16}{15}\sin(t)\cos^2(t)\right)$$

and

$$P_{TN_q} = \frac{4\cos(t)\,(16\cos^2(t)+9)}{75},$$

respectively.

Also, the partner ruled surface generated by the quasi-vectors quasi-vectors $\boldsymbol{N}_q$ and $\boldsymbol{T}$ is given by the parametric form

$$^{N_q T}\phi(t,v) = \left(-\frac{4}{5}v\sin(t), \frac{3\sin(t)}{\sqrt{16\cos^2(t)+9}} - v\cos(t), \frac{5\cos(t)}{\sqrt{16\cos^2(t)+9}} + \frac{3}{5}v\sin(t)\right).$$

Equations (09) of the partner ruled surface yield the Gaussian and mean curvatures as

$$K_2 = \frac{1296}{\Omega^2}\sin^2(t)\,(16\cos^2(t)+9)$$

$$H_2 = \frac{36}{\Omega^{3/2}}\Big[v(32\cos^3(t) - 21\sin(t) - 57\cos(t))\sqrt{16\cos^2(t)+9} + 96\cos^2(t)\sin(t)\Big]$$

respectively, where $\Omega = 256\cos^4(t)\,v^2 + 96v\sin(t)\cos(t)\sqrt{16\cos^2(t)+9} + 288v^2\cos^2(t) + 144\sin^2(t) + 81v^2$.

The striction curves and distribution parameter on the partner ruled surface $^{N_q T}\phi(t,v)$ are given by

$$\beta_{N_q T}(t) = \left(-\frac{192}{5}\frac{\cos(t)\sin^2(t)}{(16\cos^2(t)+9)^{\frac{3}{2}}}, \frac{27\sin(t)}{(16\cos^2(t)+9)^{\frac{3}{2}}}, \frac{256}{5}\frac{cos^3(t)+369\cos(t)}{(16\cos^2(t)+9)^{\frac{3}{2}}}\right)$$

and

$$P_{N_q T} = \frac{36\sin(t)}{(16\cos^2(t)+9)^{\frac{3}{2}}},$$

respectively.

Finally, Figure 1 shows the directional partner ruled surface $^{TN_q}\phi(t,v)$ (green) and the directional partner surface $^{N_q T}\phi(t,v)$ (cyan), where both the surfaces and their striction lines are drawn in three-dimensional space.

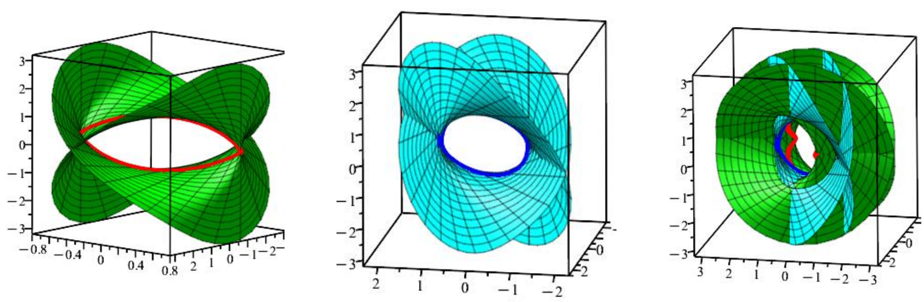

Figure 1 The partner ruled surfaces $^{TN_q}\phi(t,v)$ and $^{N_q T}\phi(t,v)$ in 3-space.

Example 2 Let us consider a curve parameterized as

$$\alpha(t) = (6t, 3t^2, t^3). \tag{31}$$

Then, the quasi-vectors of $\alpha(t)$ are given by

$$\begin{aligned} \boldsymbol{T} &= \left(\frac{2}{t^2+2}, \frac{2t}{t^2+2}, \frac{t^2}{t^2+2}\right) \\ \boldsymbol{N}_q &= \left(0, \frac{t}{\sqrt{t^2+4}}, -\frac{t^2}{t^2+2}\right) \\ \boldsymbol{B}_q &= \left(-\frac{t}{(t^2+2)\sqrt{t^2+4}}, \frac{4}{(t^2+2)\sqrt{t^2+4}}, \frac{2t}{(t^2+2)\sqrt{t^2+4}}\right). \end{aligned} \tag{32}$$

Substituting equation (32) into equation (10), the partner ruled surface formed by the quasi-vectors $\boldsymbol{T}$ and $\boldsymbol{N}_q$ is parametrized as

$$^{\boldsymbol{TN}_q}\phi(t,v) = \left(\frac{2}{t^2+2}, \frac{2t}{t^2+2} + \frac{vt}{\sqrt{t^2+4}}, \frac{t^2}{t^2+2} - \frac{2v}{\sqrt{t^2+4}}\right).$$

Substituting equation (32) into equation (11), the partner ruled surafce generated by the $\boldsymbol{N}_q$ and $\boldsymbol{T}$ quasi-vectors is parametrized as

$$^{\boldsymbol{N}_q\boldsymbol{T}}\phi(t,v) = \left(\frac{2v}{t^2+2}, \frac{t}{\sqrt{t^2+4}} + \frac{2vt}{t^2+2}, -\frac{2}{\sqrt{t^2+4}} + \frac{t^2}{t^2+2}\right).$$

Finally, Figure 2 shows the directional partner ruled surface $^{\boldsymbol{TN}_q}\phi(t,v)$ (purple) and the directional partner surface $^{\boldsymbol{N}_q\boldsymbol{T}}\phi(t,v)$ (yellow), where both the surfaces and their striction lines are drawn in three-dimensional space.

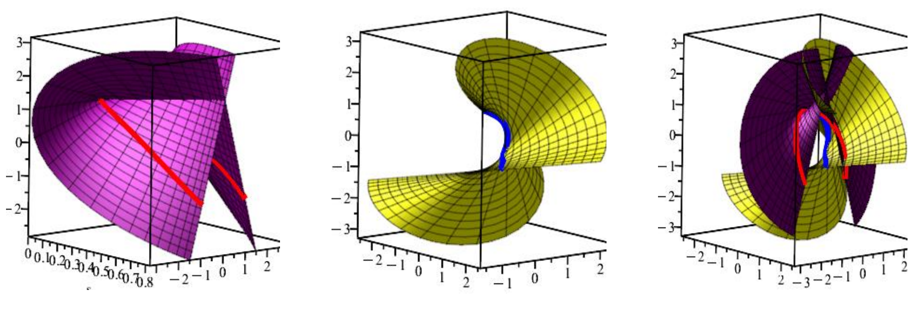

Figure 2 The partner ruled surfaces $^{\boldsymbol{TN}_q}\phi(t,v)$ and $^{\boldsymbol{N}_q\boldsymbol{T}}\phi(t,v)$ in 3-space.

The depiction of all partner ruled surfaces has provided using the Maple application.

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Chapter 2

A Partner Ruled Surface with Q-Frame in 3-Dımensional Space

Başak YAĞBASAN[1], Aybüke EKİCİ COŞKUN[2]

ABSTRACT

In this study, q-vectors, q-derivative equations and κ, τ and η curvatures of the curve for a q-framed curve in 3-dimensional space are mentioned. The surface formed by moving a straight line along every point on a curve is called a ruled surface. The general equation of the partner ruled surface in Euclidean 3-space, formed using the q-frame vectors $\boldsymbol{T}$ and $\boldsymbol{B}_q$, is given. Using the general equation of the surface, the unit normal vector field, the first and second fundamental form coefficients, Gaussian and mean curvatures, distribution parameters and contraction lines are obtained. Finally, in addition to theoretical calculations, example was given and the shapes of the surfaces are plotted.

Keywords: Euclid 3-space, q-frame, ruled surface, partner ruled surface.

3-BOYUTLU UZAYDA Q-ÇATILI BİR PARTNER REGLE YÜZEYİ

ÖZET

Bu çalışmada 3-boyutlu uzayda q-çatılı bir eğri için q-vektörleri, q-türev denklemleri ve eğrinin κ, τ ve η eğriliklerinden bahsedilmiştir. Bir doğrunun, bir eğri üzerindeki her noktada hareket ettirilmesiyle oluşan yüzeye regle yüzey denir. Öklid 3-uzayında partner regle yüzeyin q-çatı vektörlerinden $\boldsymbol{T}$ ve $\boldsymbol{B}_q$ kullanılarak oluşturulan genel denklemi verilmiştir. Yüzeyin genel denklemi kullanılarak, birim normal vektör alanı, birinci ve ikinci temel form katsayıları, Gauss ve ortalama eğrilikler, dağılma parametreleri ve striksiyon çizgileri elde edilmiştir. Son olarak teorik hesaplara ek olarak örnek verilmiş ve yüzeylerin şekilleri çizdirilmiştir.

Anahtar Kelimeler: Öklid 3-uzayı, q-çatısı, regle yüzey, partner regle yüzey

[1] Eskişehir Osmangazi University, Department of Mathematics and Computer Science, Eskişehir.
bskyagbasan@gmail.com, 05315724143
ORCID NO: 0000-0003- 4067-3034
[2] Eskişehir Osmangazi University, Department of Mathematics and Computer Science, Eskişehir.
aybkekici@gmail.com, 05383501819
ORCID NO: 0000-0002-5630-2900

INTRODUCTION

The analysis of some surfaces, the theoretical foundations of which were laid in classical studies, has been enriched by modern approaches (Gray, 1993). The concept of the quasi-normal vector was initially formulated within the $\mathbb{E}^3$ space (Coquillart, 1987). This vector subsequently served as the foundation for introducing an alternative frame structure, specifically the adapted q-frame, along a given space curve (Dede et al., 2015). Many contributions have been made to the literature using the q-frame. For example, new characterizations and reconstructions of surfaces using q-frame vectors in $\mathbb{E}^3$ space are noteworthy (Ekici, et al., 2021; Dede et al., 2024). The systematic study of surfaces generated by a one-parameter family of straight lines, or ruled surfaces, finds its origin in the work of Monge (Monge, 1780). These surfaces and their associated mathematical apparatus—including foundational definitions, key theorems, and essential concepts—are fundamental to the study of differential geometry, playing a significant role across theoretical investigations in both Euclid and non-Euclid space settings (Izumiya and Takeuchi, 2003). In the literature, studies on ruled surfaces have been conducted using different frame structures not only in Euclidean space but also in Galilean and Minkowski space (Alegre et al., 2010; Dede and Ekici, 2011; Dede and Ekici, 2016; Ünlütürk et al., 2016; Kaymanlı et al., 2020; Ekici et al., 2020; Kaymanlı et al., 2022; Li et al., 2023). Additionally, investigating the concept of partner ruled surfaces and examining the simultaneous properties of this surface using Darboux and alternative frames is an important focus (Ouarab, et al., 2018; Ouarab, 2021; Li et al., 2022; Cengiz, 2025). Moreover, examples of surfaces created using curves and their frames include ruled, tube, and canal surfaces. Many studies have been conducted on these surfaces. (Ekici, et al., 2017; Kızıltuğ et al., 2019; Yağbasan et al., 2023; Ekici and Akça, 2023; Yağbasan and Ekici, 2025). In this study, the differential geometric properties of partner ruled surfaces created with the help of $\boldsymbol{T}$ and $\mathbf{B}_q$ q-frame vectors are discussed.

PRELIMINARIES

Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be two vectors in $\mathbb{E}^3$. Here, the known inner product is calculated with $<\boldsymbol{V},\boldsymbol{W}>$ and the norm of the vector is calculated as $\|\boldsymbol{V}\| = \sqrt{<\boldsymbol{V},\boldsymbol{V}>}$. The vector product is expressed as

$$\boldsymbol{V}\wedge\boldsymbol{W} = (v_2w_3 - v_3w_2)\boldsymbol{e}_1 - (v_3w_1 - v_1w_3)\boldsymbol{e}_2 + (v_1w_2 - v_2w_1)\boldsymbol{e}_3$$

where $\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 = \boldsymbol{e}_3, \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 = \boldsymbol{e}_1, \boldsymbol{e}_3 \wedge \boldsymbol{e}_2 = -\boldsymbol{e}_1$ (Gray, 1993). The q-frame $\{\boldsymbol{T}(s), \mathbf{N}_q(s), \boldsymbol{B}_q(s), \mathbf{k}\}$ is given by

$$\boldsymbol{T}(s)=\frac{\alpha'}{||\alpha'||},\qquad \mathbf{N}_q(s)=\frac{\boldsymbol{T}\wedge\mathbf{k}}{||\boldsymbol{T}\wedge\mathbf{k}||},\ \ \mathbf{B}_q(s)=\boldsymbol{T}(s)\wedge\mathbf{N}_q(s) \tag{1}$$

where the projection vector $\mathbf{k}=(1,0,0)$ is chosen for ease of operation (Dede et al., 2015). The q-equations of variation for the $a(s)$ curve given by the arc parameter are

$$\begin{bmatrix}\boldsymbol{T}'\\ \boldsymbol{N}_q'\\ \boldsymbol{B}_q'\end{bmatrix}=\begin{bmatrix}0&\kappa&\tau\\ -\kappa&0&\eta\\ -\tau&-\eta&0\end{bmatrix}\begin{bmatrix}\boldsymbol{T}\\ \boldsymbol{N}_q\\ \boldsymbol{B}_q\end{bmatrix} \tag{2}$$

where the functions

$$\kappa=\langle\boldsymbol{T}',\boldsymbol{N}_q\rangle,\qquad \tau=\langle\boldsymbol{T}',\boldsymbol{B}_q\rangle,\qquad \eta=\langle\boldsymbol{N}_q',\boldsymbol{B}_q\rangle \tag{3}$$

respectively (Dede et al., 2015; Yağbasan et al., 2023). The parametric equation of ruled surface $\psi(s,u)$ is given as

$$\psi(s,u)=\alpha(s)+u.\mathrm{X}(s) \tag{4}$$

where $\alpha(s)$ is a curve and $\mathrm{X}(s)$ is a generator vector (Monge, 1780; Okur et al., 2021; Ouarab, 2021). The distribution parameter of the ruled surface is identified by

$$P_X=\frac{\det(\alpha_s,X,X_s)}{\langle X_s,X_s\rangle}. \tag{5}$$

The striction point on the ruled surface is the foot of the common perpendicular line successive rulings on the main ruling. It is given as

$$\beta_X(s)=\alpha(s)-\frac{\langle\alpha_s,X_s\rangle}{\langle X_s,X_s\rangle}\mathrm{X}(s) \tag{6}$$

(Okur et al., 2021; Ouarab, 2021). The coefficients of the first fundamental form are defined as

$$e=\langle\psi_s,\psi_s\rangle, f=\langle\psi_s,\psi_u\rangle, g=\langle\psi_u,\psi_u\rangle \text{ and } \mathrm{w}=eg-f^2 \tag{7}$$

(Do Carmo, 1976; Gray, 1993). Then the unit normal vector field of a surface is defined as

$$\boldsymbol{\mathcal{N}}=\frac{\psi_s\wedge\psi_u}{||\psi_s\wedge\psi_u||}. \tag{8}$$

The coefficients of its second fundamental form of a surface are defined as

$$l=\langle\psi_{ss},\boldsymbol{\mathcal{N}}\rangle,\qquad m=\langle\psi_{su},\boldsymbol{\mathcal{N}}\rangle \text{ and } n=\langle\psi_{uu},\boldsymbol{\mathcal{N}}\rangle. \tag{9}$$

The Gaussian and mean curvatures of the surface are typically expressed as

$$\mathcal{K}=\frac{ln-m^2}{eg-f^2} \text{ and } \mathcal{H}=\frac{lg-2mf+ne}{eg-f^2} \tag{10}$$

respectively (Do Carmo, 1976; Gray, 1993).

Partner Ruled Surfaces with $\boldsymbol{T}$ and $\boldsymbol{B_q}$ q-vectors in 3-dimensional Space

Let the q-frame associated with the curve $\alpha(s)$ on the surface $\psi(s,u)$ be designated by the orthonormal basis $\{\boldsymbol{T}(s), \boldsymbol{N}_q(s), \boldsymbol{B}_q(s)\}$. Consequently, the partner ruled surfaces generated by the expression

$$\psi^{\boldsymbol{TB}_q}(s,u) = \boldsymbol{T}(s) + u\boldsymbol{B}_q(s) \tag{11}$$

is termed $\psi^{\boldsymbol{TB}_q}$ partner ruled surfaces with respect to the q-frame of the curve $\alpha(s)$ on the surface $\psi(s,u)$.

Theorem 1 Consider a $\boldsymbol{TB}_q$ partner ruled surface parameterized by $\psi^{\boldsymbol{TB}_q}(s,u)$ within Euclidean three-space, $\mathbb{E}^3$. The unit normal vector field of this surface, denoted $\boldsymbol{\mathcal{N}}_1(s,u)$, is determined to be

$$\boldsymbol{\mathcal{N}}_1(s,u) = \frac{(\kappa - u\eta)\boldsymbol{T}(s) + u\tau\boldsymbol{N}_q(s)}{\sqrt{(\kappa - u\eta)^2 + u^2\tau^2}}.$$

Proof The first partial derivatives of $\psi^{\boldsymbol{TB}_q}(s,u)$, defined in equation (11) with respect to s and u, are determined by

$$\psi_s^{\boldsymbol{TB}_q} = -u\tau\boldsymbol{T}(s) + (\kappa - u\eta)\boldsymbol{N}_q(s) + \tau\boldsymbol{B}_q(s) \tag{12}$$

and

$$\psi_u^{\boldsymbol{TB}_q} = \boldsymbol{B}_q(s). \tag{13}$$

Then, second order partial derivatives of $\psi^{\boldsymbol{TB}_q}(s,u)$, are given as

$$\begin{aligned} \psi_{ss}^{\boldsymbol{TB}_q} &= (-\kappa^2 - \tau^2 - u\tau' + u\kappa\eta)\boldsymbol{T}(s) + (\kappa' - \tau\eta - u\kappa\tau - u\eta')\boldsymbol{N}_q(s) \\ &\quad +(\kappa\eta + \tau' - u\tau^2 - u\eta^2)\boldsymbol{B}_q(s) \\ \psi_{su}^{\boldsymbol{TB}_q} &= -\tau\boldsymbol{T}(s) - \eta\boldsymbol{N}_q(s) \\ \psi_{uu}^{\boldsymbol{TB}_q} &= 0. \end{aligned} \tag{14}$$

The unit normal vector field $\boldsymbol{\mathcal{N}}_1(s,u)$ of this surface is orthogonal to the partial derivatives $\psi_s^{\boldsymbol{TB}_q}$ and $\psi_u^{\boldsymbol{TB}_q}$ of the surface $\psi^{\boldsymbol{TB}_q}(s,u)$. Then the unit normal vector field $\boldsymbol{\mathcal{N}}_1(s,u)$ of the partner ruled surface is obtained as

$$\boldsymbol{\mathcal{N}}_1(s,u) = \frac{(\kappa - u\eta)\boldsymbol{T}(s) + u\tau\boldsymbol{N}_q(s)}{\sqrt{(\kappa - u\eta)^2 + u^2\tau^2}}. \tag{15}$$

Theorem 2 Let be a partner ruled surface to q-frame with parametrization $\psi^{\boldsymbol{TB}_q}(s,u)$ in $\mathbb{E}^3$. Gaussian curvature $\mathcal{K}_1$ and mean curvature $\mathcal{H}_1$ of partner ruled surface with unit normal vector field are obtained as

$$\mathcal{K}_1 = \frac{-\kappa^2\tau^2}{(u^2\tau^2 + (\kappa - u\eta)^2)^2}$$

and

$$\mathcal{H}_1 = \frac{-(\kappa^2(\kappa - 2u\eta) - \kappa\tau^2 - u(\kappa'\tau + \kappa\tau') + u^2(\kappa\eta^2 + \kappa\tau^2 - \tau'\eta + \tau\eta'))}{(u^2\tau^2 + (\kappa - u\eta)^2)^{3/2}}$$

respectively.

Proof Using equation (15), by substituting (12) and (13) into equation (9), the coefficients of the first fundamental form for the partner ruled surface

$$\begin{aligned} e_1 &= u^2\tau^2 + \kappa^2 - 2u\kappa\eta + u^2\eta^2 + \tau^2 \\ f_1 &= \tau \\ g_1 &= 1 \\ w_1 &= u^2\tau^2 + (\kappa - u\eta)^2 \end{aligned} \tag{16}$$

are subsequently derived. Equations (8), (14) and (15) lead to the coefficients of the second fundamental form of the partner ruled surface with the unit vector field in $\mathbb{E}^3$ obtained as,

$$\begin{aligned} l_1 &= \frac{-(\kappa^3 - 2u\kappa^2\eta + \kappa\tau^2 - u(\kappa'\tau - \kappa\tau') + u^2(\kappa\eta^2 + \kappa\tau^2 - \tau'\eta + \tau\eta'}{\sqrt{u^2\tau^2 + (\kappa - v\eta)^2}} \\ m_1 &= \frac{\kappa\tau}{\sqrt{u^2\tau^2 + (\kappa - u\eta)^2}} \\ n_1 &= 0. \end{aligned} \tag{17}$$

Substituting equations (16) and (17) into equation (10) implies that Gaussian and mean curvatures with respect to $\boldsymbol{\mathcal{N}}_1(s,u)$ following as

$$\mathcal{K}_1 = \frac{-\kappa^2\tau^2}{(u^2\tau^2 + (\kappa - u\eta)^2)^2}$$

and

$$\mathcal{H}_1 = \frac{-(\kappa^3 - 2u\kappa^2\eta - \kappa\tau^2 - u(\kappa'\tau - \kappa\tau') + u^2(\kappa\eta^2 + \kappa\tau^2 - \tau'\eta + \tau\eta'))}{(u^2\tau^2 + (\kappa - u\eta)^2)^{3/2}}.$$

Theorem 3 The striction curves and distribution parameter on the partner ruled surface using the $\boldsymbol{T}'(s), \boldsymbol{B}_q(s)$ and $\boldsymbol{B}_q'(s)$ are given by

$$\beta_{\boldsymbol{TB}_q}(s) = \boldsymbol{T}(s) + \frac{\kappa\eta}{\tau^2+\eta^2}\boldsymbol{B}_q(s)$$

and

$$P_{\boldsymbol{TB}_q} = -\frac{\kappa\tau}{\tau^2 + \eta^2}$$

respectively.

Proof If the equation (2) is used in equations (5) and (6), the striction line and the distribution parameter of the $\psi^{\boldsymbol{TB}_q}(s,u)$ surface are explicitly omitted, respectively.

Partner Ruled Surfaces with T and B_q q-vectors in 3-dimensional Space

Let the q-frame associated with the curve $\alpha(s)$ on the surface $\psi(s,u)$ be designated by the orthonormal basis $\{\boldsymbol{T}(s), \boldsymbol{N}_q(s), \boldsymbol{B}_q(s)\}$. Consequently, the ruled surfaces generated by the expression

$$\psi^{\boldsymbol{B}_q\boldsymbol{T}}(s,u) = \boldsymbol{B}_q(s) + u\boldsymbol{T}(s) \tag{18}$$

is termed $\psi^{\boldsymbol{T}\boldsymbol{B}_q}$ partner ruled surfaces with respect to the q-frame of the curve $\alpha(s)$ on the surface $\psi(s,u)$.

Theorem 4 Let be a partner ruled surface q-frame with parametrization $\psi^{\boldsymbol{B}_q\boldsymbol{T}}(s,u)$ in 3-dimensional space. Unit normal vector field $\boldsymbol{\mathcal{N}}_2(s,u)$ of the partner ruled surface in $\mathbb{E}^3$ is obtained as

$$\boldsymbol{\mathcal{N}}_2(s,u) = \frac{u\tau\boldsymbol{N}_q(s) + (\eta - u\kappa)\boldsymbol{B}_q(s)}{\sqrt{u^2\tau^2 + (\eta - u\kappa)^2}}.$$

Proof As a necessary step for the proof, the first order partial derivatives of the partner ruled surface defined by equation (18) are computed to yield the expression

$$\psi_s^{\boldsymbol{B}_q\boldsymbol{T}} = -\tau\boldsymbol{T}(s) + (-\eta + u\kappa)\boldsymbol{N}_q(s) + u\tau\boldsymbol{B}_q(s) \tag{19}$$

and

$$\psi_u^{\boldsymbol{B}_q\boldsymbol{T}} = \boldsymbol{T}(s). \tag{20}$$

Then, second order partial derivatives of $\psi^{\boldsymbol{B}_q\boldsymbol{T}}(s,u)$, are given as

$$\begin{aligned} \psi_{ss}^{\boldsymbol{B}_q\boldsymbol{T}} &= (\kappa\eta - u\kappa^2 - u\tau^2 - \tau')\boldsymbol{T}(s) + (u\kappa' - u\tau\eta - \kappa\tau - \eta')\boldsymbol{N}_q(s) \\ &\quad +(u\kappa\eta + u\tau' - \tau^2 - \eta^2)\boldsymbol{B}_q(s) \\ \psi_{su}^{\boldsymbol{B}_q\boldsymbol{T}} &= \kappa\boldsymbol{N}_q(s) + \tau\boldsymbol{B}_q(t) \\ \psi_{uu}^{\boldsymbol{B}_q\boldsymbol{T}} &= 0. \end{aligned} \tag{21}$$

The unit normal vector field $\boldsymbol{\mathcal{N}}_2(s,u)$ of this surface is orthogonal to the partial derivatives $\psi_s^{\boldsymbol{B}_q\boldsymbol{T}}$ and $\psi_u^{\boldsymbol{B}_q\boldsymbol{T}}$ of the surface $\psi^{\boldsymbol{B}_q\boldsymbol{T}}(s,u)$. Then the unit normal vector field $\boldsymbol{\mathcal{N}}_2(s,u)$ of partner ruled surface is obtained as

$$\boldsymbol{\mathcal{N}}_2(s,u) = \frac{u\tau\boldsymbol{N}_q(s) + (\eta - u\kappa)\boldsymbol{B}_q(s)}{\sqrt{u^2\tau^2 + (\eta - u\kappa)^2}}. \tag{22}$$

Theorem 5 Let be a partner ruled surface q-frame with parametrization $\psi^{\boldsymbol{B}_q\boldsymbol{T}}(s,u)$ in 3-dimensional space. Gaussian curvature $\mathcal{K}_2$ and mean curvature $\mathcal{H}_2$ of the partner ruled surface with unit normal vector field $\boldsymbol{\mathcal{N}}_2(s,u)$ in $\mathbb{E}^3$ are obtained as

$$\mathcal{K}_2 = \frac{\tau^2\eta^2}{(u^2\tau^2 + (\eta - u\kappa)^2)^2}$$

and

$$\mathcal{H}_2 = \frac{-(\eta^3 - \tau^2\eta - 2u\kappa\eta^2 - u(\tau'\eta - \tau\eta') + u^2(\tau^2\eta + \kappa^2\eta - \kappa'\tau + \kappa\tau'))}{(u^2\tau^2 + (\eta - u\kappa)^2)^{3/2}}$$

respectively.

Proof Using equation (22), by substituting (12) and (13) into equation (9), the coefficients of the first fundamental form for the partner ruled surfaces

$$\begin{aligned} e_2 &= \tau^2 + \eta^2 - 2u\kappa\eta + u^2\kappa^2 + u^2\tau^2 \\ f_2 &= -\tau \\ g_2 &= 1 \\ w_2 &= (\eta - u\kappa)^2 + u^2\tau^2 \end{aligned} \tag{23}$$

are subsequently derived. Equations (8), (21) and (22) lead to the coefficients of the second fundamental form of this surface with the unit vector field $\boldsymbol{\mathcal{N}}_2(s,u)$ in $\mathbb{E}^3$ obtained as,

$$\begin{aligned} l_2 &= \frac{-(\tau^2\eta + \eta^3 - 2u\kappa\eta^2 - u(\tau'\eta - \tau\eta') + u^2(\tau^2\eta + \kappa^2\eta - \kappa'\tau + \kappa\tau'}{\sqrt{u^2\tau^2 + (\eta - u\kappa)^2}} \\ m_2 &= \frac{\tau\eta}{\sqrt{u^2\tau^2 + (\eta - u\kappa)^2}} \\ n_2 &= 0. \end{aligned} \tag{24}$$

Substituting equations (23) and (24) into equation (10) implies that Gaussian and mean curvatures with respect to $\boldsymbol{\mathcal{N}}_2(s,u)$ following as

$$\mathcal{K}_2 = \frac{-\tau^2\eta^2}{(u^2\tau^2 + (\eta - u\kappa)^2)^2}$$

and

$$\mathcal{H}_2 = \frac{-(\eta^3 - \tau^2\eta - 2u\kappa\eta^2 - u(\tau'\eta - \tau\eta') + u^2(\tau^2\eta + \kappa^2\eta - \kappa'\tau + \kappa\tau'))}{(u^2\tau^2 + (\eta - u\kappa)^2)^{3/2}}.$$

Theorem 6 The striction curves and distribution parameter on the partner ruled surface for the $\psi^{\boldsymbol{B}_q\boldsymbol{T}}(s,u)$ space are given by

$$\beta_{\boldsymbol{B}_q\boldsymbol{T}}(s) = \frac{\kappa\eta}{\kappa^2 + \tau^2}\boldsymbol{T}(s) + \boldsymbol{B}_q(s)$$

and

$$P_{\boldsymbol{B}_q\boldsymbol{T}} = \frac{\tau\eta}{\tau^2 + \eta^2}$$

respectively.

Proof If the equation (2) is used in equations (5) and (6), the striction line and the distribution parameter of the $\psi^{\boldsymbol{B}_q\boldsymbol{T}}(s,u)$ surface are explicitly omitted, respectively.

Example 1 Let $\alpha(s)$ be a centre curve with q-frame of partner ruled surface in $\mathbb{E}^3$ such as

$$\alpha(s) = \frac{1}{\sqrt{5}}\left(s\sqrt{1+s^2}, 2s, ln\left(s+\sqrt{1+s^2}\right)\right). \quad (25)$$

From $\left|\left|\alpha(s)\right|\right| = 1$, it is easy to see that q-vectors are given as

$$\boldsymbol{T} = \frac{1}{\sqrt{5}}\left(\frac{t}{\sqrt{1+s^2}}, 2, \frac{1}{\sqrt{1+s^2}}\right), \qquad \boldsymbol{N}_q = \frac{1}{\sqrt{5+4s^2}}\left(0,1,-2\sqrt{1+s^2}\right)$$
$$\boldsymbol{B}_q = \frac{1}{\sqrt{5}\sqrt{5+4s^2}}\left(-\frac{5+4s^2}{(5+5s^2)\sqrt{1+s^2}}, 2s, \frac{s}{\sqrt{1+s^2}}\right) \quad (26)$$

and from equation (3), q-curvatures are given as

$$\kappa = \frac{2s}{(1+s^2)\sqrt{5}\sqrt{5+4s^2}}, \qquad \tau = -\frac{1}{(1+s^2)\sqrt{5+4s^2}}$$
$$\eta = -\frac{2s^2}{(1+s^2)(5+4s^2)\sqrt{5}}.$$

Substituting equations (25) and (26) into equation (11), the partner ruled surface formed by the q-vectors $\boldsymbol{T}$ and $\boldsymbol{B}_q$ is parametrized as

$$\psi^{\boldsymbol{TB}_q}(s,u) = \frac{1}{\sqrt{5}}\left(\frac{s}{\sqrt{1+s^2}}, 2, \frac{1}{\sqrt{1+s^2}}\right) + \frac{u}{\sqrt{5}\sqrt{5+4s^2}}\left(-\frac{5+4s^2}{\sqrt{1+s^2}}, 2s, \frac{s}{\sqrt{1+s^2}}\right). \quad (27)$$

Then the unit normal vector field in equation (28) of this surface is given as

$$\boldsymbol{\mathcal{N}}_1(s,u) = \frac{1}{\sqrt{5}\sqrt{1+s^2}\sqrt{\mathcal{A}}}\left(2s^2\left(us+\sqrt{5+4s^2}\right), \frac{4t\sqrt{5+4s^2}}{\sqrt{1+s^2}}(s^2+1) + u\sqrt{1+s^2}(4s^2-5), 2(6us^2+5u+s\sqrt{5+4s^2})\right) \quad (28)$$

where $\mathcal{A} = 4s^4(u^2+4) + 8us^3\sqrt{5+4s^2} + 20s^2(u^2+1) + 25u^2$. For equation (28), the Gaussian and mean curvatures in equations (29) of the partner ruled surface are given as

$$\mathcal{K}_1 = -\frac{20s^2(5+4s^2)^2}{\mathcal{A}^2}$$
$$\mathcal{H}_1 = -\frac{2}{\mathcal{A}^{3/2}\sqrt{5+4s^2}}(16s^5 + u^2s(24s^4+90s^2+75) + u\sqrt{5+4s^2}(28s^4+45s^2+25) - 25s) \quad (29)$$

respectively. The striction curves and distribution parameter on the partner ruled surface

$$\begin{aligned}\beta_{\boldsymbol{TB}_q}(s) &= \Bigg(\frac{1}{\sqrt{5}\sqrt{1+s^2}}\left(s-\frac{4s^3(5+4s^2)}{25+4s^4+20s^2}\right),\\ &+\frac{2}{\sqrt{5}}\left(1+\frac{4s^4}{25+4s^4+20s^2}\right),\\ &+\frac{1}{\sqrt{5}\sqrt{1+s^2}}\left(1+\frac{4s^4}{25+4s^4+20s^2}\right)\Bigg)\end{aligned}$$

and

$$P_{\boldsymbol{TB}_q}=\frac{2s\sqrt{5}(5+4s^2)}{25+4s^4+20s^2}$$

respectively.

Substituting equations (25) and (26) into equation (18), the partner ruled surface formed by the q-vectors $\boldsymbol{B}_q$ and $\boldsymbol{T}$ in $\mathbb{E}^3$ is parametrized as

$$\begin{aligned}\psi^{\boldsymbol{B}_q\boldsymbol{T}}(s,u) = &\frac{1}{\sqrt{5}\sqrt{5+4s^2}}\left(-\frac{1}{\sqrt{1+s^2}},2s,\frac{s}{\sqrt{1+s^2}}\right)\\ &+\frac{u}{\sqrt{5}}\left(-\frac{s}{\sqrt{1+s^2}},2,\frac{1}{\sqrt{1+s^2}}\right).\end{aligned} \tag{30}$$

Then the unit normal vector field in equation (31) of this surface is given as

$$\begin{aligned}\boldsymbol{\mathcal{N}}_2(s,u) = &\frac{1}{\sqrt{5}\sqrt{5+4s^2}\sqrt{\mathcal{B}}}\Bigg(\frac{2s(5+4s^2)\left(s+u\sqrt{5+4s^2}\right)}{\sqrt{1+s^2}},\\ &-4s^3+u\sqrt{5+4s^2}(4s^2+5),\\ &\frac{2(-s^3+u\sqrt{5+4s^2}(4s^2+5))}{\sqrt{1+s^2}}\Bigg)\end{aligned} \tag{31}$$

where $\mathcal{B}=4s^4+8us^3\sqrt{5+4s^2}+40u^2s^2+16u^2s^4+25u^2$. For equation (31), the Gaussian and mean curvatures in equation (32) of partner ruled surface are given as

$$\begin{aligned}\mathcal{K}_2 = &-\frac{20s^4(5+4s^2)}{\mathcal{B}^2}\\ \mathcal{H}_2 = &-\frac{2}{\mathcal{B}^2\sqrt{5+4s^2}}(s^2(50s^2+25-4s^4)+4u^2s(16s^5+60s^3+75\\ &+2u\sqrt{5+4s^2}(6s^5+25s+35s^3)+125u^2)\end{aligned} \tag{32}$$

respectively. The striction curves and distribution parameter on the partner ruled surface

$$\beta_{\boldsymbol{B}_q\boldsymbol{T}}(s) = \left(\frac{1}{\sqrt{5}\sqrt{1+s^2}}\left(-\sqrt{5+4s^2}+\frac{4s^4}{(5+4s^2)^{3/2}}\right),\right.$$
$$+\frac{2}{\sqrt{5}}\left(\frac{s}{\sqrt{5+4s^2}}+\frac{4s^3}{(5+4s^2)^{3/2}}\right),$$
$$\left.+\frac{1}{\sqrt{5}\sqrt{1+s^2}}\left(\frac{s}{\sqrt{5+4s^2}}+\frac{4s^3}{(5+4s^2)^{3/2}}\right)\right)$$

and

$$P_{\boldsymbol{B}_q\boldsymbol{T}} = \frac{2s^2\sqrt{5}}{(5+4s^2)^{3/2}}$$

respectively.

Finally, a directional partner ruled surface is drawn in 3-space, as shown in Figure 1. Here, the red and blue curves in Figure 1 are the striction curves of the $\psi^{\boldsymbol{T}\boldsymbol{B}_q}(s,u)$ and $\psi^{\boldsymbol{B}_q\boldsymbol{T}}(s,u)$ surfaces, respectively.

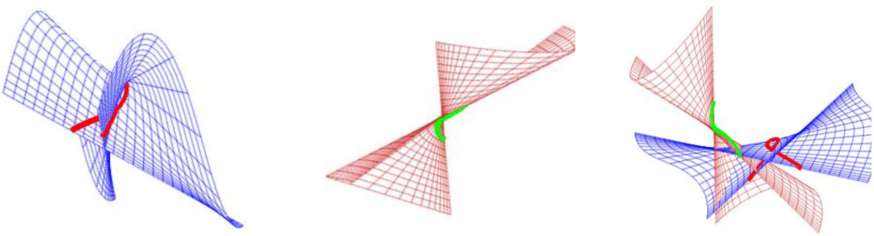

Figure 1 Directional partner ruled surface in 3-space.

The visualization of all surfaces are given with using Maple programme.

CONCLUSION

In this study, the parameterization of the directional partner ruled surface generated with q-vectors $\boldsymbol{T}$ and $\boldsymbol{B}_q$ is given. The unit normal vector field, Gaussian curvatures, mean curvatures, and striction curves and distribution parameter of this surface is obtained. An example is given and plotted in 3-space.

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Chapter 3

On the Quasi Ruled Hypersurfaces in Euclidean 4-Space

Gül UĞUR KAYMANLI[1]

INTRODUCTION

In order to gain a deeper understanding of the phenomena occurring in our surroundings, it is essential to study surfaces. Consequently, developing a clear framework for the construction of surfaces i fundamental in differential geometry. Owing to their structural efficiency and the relative simplicity of their geometric construction, ruled surfaces represent a particularly important and widely studied class of surfaces. A ruled surface is represented by special type of surface generated by the continuous motion of a straight line, referred to as a ruling, along a curve.

Within this framework, the geometry of ruled surfaces has been studied in [1], [5], [8], [10]-[12] from various perspectives, leading to a rich body of results concerning their intrinsic and extrinsic properties. In 2021, the authors worked on 2-Ruled hypersurfaces in four dimensional Euclidean space and examined its geeometric properties by studying Gauss map in [2] while Wang studied it complex space forms of shape operator in [13]. After Wang investigated it, he analysed nonflat complex space forms in [14]. In four dimensional space, surfaces were explored in [15], [16]. Most recently, ruled hypersurfaces both in Euclidean [7] and Minkowski [17] spaces investigated by focusing on polarized light wave.

In this section, we study the hypersurface induced by the quasi-type vector fields, quasi-tangent, and second binormal vectors in four-dimensional Euclidean space. A parametric formulation of the hypersurface is presented, followed by an analysis of its fundamental geometric characteristics. In particular, we derive the first and second fundamental forms, the associated shape operator, the Gaussian and mean curvatures, to determine whether the hypersurface is minimal and flat.

[1] Department of Mathematics, Çankırı Karatekin University, Çankırı, Turkey
gulugurk@karatekin.edu.tr
ORCID ID: 0000-0003-4932-894X[1]

QUASI FRAME AND HYPERSURFACES IN E^4

Let E^4 be four-dimensional Euclidean space and $\beta(s)$ be the curve with frame $\{T_q, N_q, B_q, C_q\}$ called quasi tangent, quasi normal, first quasi binormal and second quasi binormal vector fields respectively. These quasi frame vectors are calculated as

$$T_q(s) = \frac{\beta'(s)}{\| \beta'(s) \|}$$

$$N_q(s) = \frac{T_q(s) \wedge k_x \wedge k_y}{\| T_q(s) \wedge k_x \wedge k_y \|}$$

$$C_q(s) = \frac{\beta'(s) \wedge N_q(s) \wedge \beta'''(s)}{\| \beta'(s) \wedge N_q(s) \wedge \beta'''(s) \|}$$

$$B_q(s) = C_q(s) \wedge T_q(s) \wedge N_q(s)$$

where the q-curvatures are

$$k_1 = \frac{<T_q'(s), N_q(s)>}{\| \alpha' \|}$$

$$k_2 = \frac{<T_q'(s), B_q(s)>}{\| \alpha' \|}$$

$$k_3 = \frac{<N_q'(s), B_q(s)>}{\| \alpha' \|}$$

$$k_4 = \frac{<B_q'(s), C_q(s)>}{\| \alpha' \|}$$

and $k_z$and $k_t$are unit standard vectors in E^4 [3], [4]. The derivation formula is written as

$$\begin{bmatrix} T_q' \\ N_q' \\ B_q' \\ C_q' \end{bmatrix} = \| \alpha'(s) \| \begin{bmatrix} 0 & k_1 & k_2 & 0 \\ -k_1 & 0 & k_3 & 0 \\ -k_2 & -k_3 & 0 & k_4 \\ 0 & 0 & -k_4 & 0 \end{bmatrix} \begin{bmatrix} T_q \\ N_q \\ B_q \\ C_q \end{bmatrix}.$$

Let $M \subset E^4$be a hypersurface parametrized by

$$\phi: U \subset \mathbb{R}^3 \longrightarrow E^4$$

$$(t,u,v) \longmapsto \phi(t,u,v)$$

$$= (\phi_1(t,u,v), \phi_2(t,u,v), \phi_3(t,u,v), \phi_4(t,u,v))$$

The image $\phi(U)$defines a hypersurface in E^4 if and only if the vectors $\{\phi_t, \phi_u, \phi_v\}$ are linearly independent at each point.

The unit normal vector field of the hypersurface is given by

$$N = \frac{\phi_t \wedge \phi_u \wedge \phi_v}{\| \phi_t \wedge \phi_u \wedge \phi_v \|}.$$

The first fundamental form of the hypersurface is defined as

$$I: \mathcal{X}(M) \times \mathcal{X}(M) \to C^{\infty}(M, \mathbb{R})$$

$$(X, Y) \to I(X, Y) = \langle X, Y \rangle$$

where $\mathcal{X}(M)$ be the space of smooth vector fields on M. The coefficients of the first fundamental form are

$$a_{ij} = \langle \phi_i, \phi_j \rangle, 1 \leq i, j \leq 3,$$

and the corresponding matrix representation is

$$I = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

Let D denote the Levi-Civita connection of $\mathbb{R}^4$. Using normal vector field, the shape operator of M is defined by

$$S: \mathcal{X}(M) \to \mathcal{X}(M)$$

$$X \to S(X) = -D_X N$$

The second fundamental form is given by

$$II: \mathcal{X}(M) \times \mathcal{X}(M) \to C^{\infty}(M, \mathbb{R})$$

$$(X, Y) \to III(X, Y) =$$

$\langle S(X), Y \rangle$

The matrix representation of this fundamental form is

$$II = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix},$$

where the coefficients are

$$b_{ij} = \langle \phi_{ij}, N \rangle, 1 \leq i, j \leq 3.$$

The shape operator can be computed by

$$S = I^{-1} II.$$

The Gauss curvature is introduced as

$$K(p) = \det\ S(p).$$

The mean curvature is given

$$H(p) = \frac{1}{3} \mathrm{tr}\ S(p).$$

Theorem 2.1. The ruled surface is said to be flat if and only if $K = 0$ [6], [9].

Theorem 2.2. A ruled surface is minimal precisely when its mean curvature vanishes [6].

Ruled surfaces are the surface generated by curves and straight lines. Let $\alpha(s)$ be a space curve and $X(s)$ a direction vector field. Then the ruled surface is given by the parametrization

$$\phi(s, u) = \alpha(s) + uX(s).$$

RULED HYPERSURFACES GENERATED WITH T_q AND C_q

In this section, we consider the hypersurface generated by the quasi type vector fields $T_q$and C_q. We provide a parametric representation of the hypersurface and investigate its fundamental geometric properties, including fundamental forms, the shape operator, and curvature invariants. Motivated by the above construction, we introduce the following parametric representation of the hypersurface:

$$\mathcal{H}(u,v,w) = \alpha(u) + vT_q(u) + wC_q(u).$$

For notational simplicity, let $T_q(u), N_q(u), B_q(u)$,and $C_q(u)$simply by T_q, N_q, B_q,and C_q, respectively.

Theorem 3.1. The shape operator of the quasi ruled hypersurface

$$\mathcal{H}(u,v,w) = \alpha(u) + vT_q(u) + wC_q(u)$$

is given by

$$S_{\mathcal{H}} = \begin{bmatrix} A - wk_1k_4 & -wk_1k_4 & vk_1k_4 \\ -A - wk_1k_4(1 + w^2k_1^2 + (vk_2 - wk_4)^2) & wk_1k_4 & -vk_1k_4 \\ vk_1k_4(v^2k_1^2 + (vk_2 - wk_4)^2) & 0 & 0 \end{bmatrix}.$$

Proof The first-order partial derivatives of the function $\mathcal{H}$ are given as follows:

$$\mathcal{H}_u = \alpha'(u) + vT_q'(u) + wC_q'(u) = T_q + vk_1N_q + (vk_2 - wk_4)B_q,$$
$$\mathcal{H}_v = T_q$$
$$\mathcal{H}_w = C_q$$

respectively. The vector $\mathcal{H}_u \wedge \mathcal{H}_v \wedge \mathcal{H}_w$used in the computation of the normal vector of hypersurface is given as follows

$$\mathcal{H}_u \wedge \mathcal{H}_v \wedge \mathcal{H}_w = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ 1 & vk_1 & vk_2 - wk_4 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} e_2 & e_3 & e_4 \\ vk_1 & vk_2 - wk_4 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= (vk_2 - wk_4)e_2 - vk_1e_3$$

The norm of $\mathcal{H}_u \wedge \mathcal{H}_v \wedge \mathcal{H}_w$is calculated as

$$\| \mathcal{H}_u \wedge \mathcal{H}_v \wedge \mathcal{H}_w \| = \sqrt{(vk_2 - wk_4)^2 + v^2k_1^2}.$$

For simplicity, let

$$W = (vk_2 - wk_4)^2 + v^2k_1^2.$$

Consequently, the normal vector of the hypersurface is given by

$$N_{\mathcal{H}} = \frac{1}{\sqrt{W}}[(vk_2 - wk_4)e_2 - vk_1e_3].$$

The second-order partial derivatives of the function $\mathcal{H}$ are given as follows

$$\mathcal{H}_{uu} = (-vk_2(k_1 + k_2) + wk_2k_4)T_q + (k_1 + v(k_1' - k_2k_3) + wk_3k_4)N_q$$
$$+(k_2 + v(k_2' + k_2k_3) - wk_4')B_q + (vk_2k_4 - wk_4^2)C_q,$$
$$\mathcal{H}_{uv} = k_1N_q + k_2B_q,$$
$$\mathcal{H}_{uw} = -k_4B_q,$$
$$\mathcal{H}_{vu} = k_1N_q + k_2B_q$$
$$\mathcal{H}_{vv} = 0$$
$$\mathcal{H}_{vw} = 0$$

and

$$\mathcal{H}_{wu} = -k_4B_q,$$
$$\mathcal{H}_{wv} = 0$$
$$\mathcal{H}_{ww} = 0$$

With the help of first-order partial derivatives of the function $\mathcal{H}$, the matrix of first fundamental form is written

$$I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 + v^2k_1^2 + (vk_2 - wk_4)^2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where a_{ij} be the coefficients of first fundamental form such that

$$a_{11} = \langle \mathcal{H}_u, \mathcal{H}_u \rangle = 1 + v^2k_1^2 + (vk_2 - wk_4)^2$$
$$a_{22} = \langle \mathcal{H}_v, \mathcal{H}_v \rangle = 1$$
$$a_{33} = \langle \mathcal{H}_w, \mathcal{H}_w \rangle = 1$$

and

$$a_{12} = a_{21} = \langle \mathcal{H}_u, \mathcal{H}_v \rangle = 1$$
$$a_{13} = a_{31} = \langle \mathcal{H}_u, \mathcal{H}_w \rangle = 0$$
$$a_{23} = a_{32} = \langle \mathcal{H}_v, \mathcal{H}_w \rangle = 0.$$

Similarly, using second-order partial derivatives of the function $\mathcal{H}$, the matrix of second fundamental form is

$$II = \frac{1}{\| W \|} \begin{bmatrix} A & -wk_1k_4 & vk_1k_4 \\ -wk_1k_4 & 0 & 0 \\ vk_1k_4 & 0 & 0 \end{bmatrix}.$$

In order to find shape operator, we need $detI$ and adjoint matrix that is,

$$I^{-1} = \frac{1}{detI} I^{\star}$$

$$= \frac{1}{v^2k_1^2 + (vk_2 - wk_4)^2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 + v^2k_1^2 + (vk_2 - wk_4)^2 & 0 \\ 0 & 0 & v^2k_1 + (vk_2 - wk_4)^2 \end{bmatrix}.$$

Similarly, to complete the proof, one can find

$$S_{\mathcal{H}} = I^{-1}II$$

$$= \begin{bmatrix} A - wk_1k_4 & -wk_1k_4 & vk_1k_4 \\ -A - wk_1k_4(1 + w^2k_1^2 + (vk_2 - wk_4)^2) & wk_1k_4 & -vk_1k_4 \\ vk_1k_4(v^2k_1^2 + (vk_2 - wk_4)^2) & 0 & 0 \end{bmatrix}.$$

Theorem 3.2. The quasi ruled hypersurface $\mathcal{H}(u, v, w)$ is flat.

Proof Since the Gauss curvature of the quasi ruled hypersurface generated by T_q and C_q is

$$K_{\mathcal{H}} = \det S_{\mathcal{H}} = 0,$$

this hypersurface is flat everywhere.

Theorem 3.3. The quasi ruled hypersurface $\mathcal{H}(u, v, w)$ is minimal if and only if one of the following conditions is satisfied:

i) Either k_1 or k_3 vanishes and there exists a relation between k_2 and k_4 given by

$$\frac{k_2}{k_4} = -\frac{w}{v}.$$

ii) Either k_2 or k_4 vanishes and the equation

$$k_1 + \left(v\left(\frac{k_1}{k_4}\right)' - wk_3\right)k_4 = 0$$

is satisfied.

Proof The mean curvature of the quasi ruled hypersurface generated by T_q and C_q is

$$H_{\mathcal{H}} = \operatorname{tr} S_{\mathcal{H}}$$

$$= \frac{1}{W}[v^2k_2(k_2(\frac{k_1}{k_2})' - k_3(k_1 + k_2)) + vw(2k_2k_3k_4 + k_1^2(\frac{k_4}{k_1})') - wk_1k_4 - w^2k_3k_4^2].$$

The proof follows immediately by using above equation.

Example 3.4 We consider the curve

$$\beta(s) = \begin{bmatrix} -s\sin s - \cos s \\ s\cos s - \sin s \\ -s\sin(3s) + \frac{1}{3}\cos(3s) \\ s\cos(3s) + \frac{1}{3}\sin(3s) \end{bmatrix},$$

for $s > 0$.

Since its velocity vector is

$$\beta'(s) = (-s\cos s, -s\sin s, -3s\cos(3s), -3s\sin(3s)),$$

its norm is

$$\| \beta'(s) \| = \sqrt{s^2 + 9s^2} = \sqrt{10}\, s.$$

Therefore, the unit tangent vector is

$$T_q(s) = \frac{\beta'(s)}{\| \beta'(s) \|} = \frac{1}{\sqrt{10}}(-\cos s, -\sin s, -3\cos(3s), -3\sin(3s)).$$

Choosing the standard basis vectors $k_z = (0,0,1,0)$and $k_t = (0,0,0,1)$, we obtain

$$T_q(s) \wedge k_z \wedge k_t = \left(-\frac{1}{\sqrt{10}}\sin s, \frac{1}{\sqrt{10}}\cos s, 0, 0\right).$$

Hence, the normal vector is

$$N_q(s) = (-sins, coss, 0,0).$$

After computing $\beta'''(s)$ and simplifying, we obtain

$$C_q(s) = (0,0,\cos(3s), \sin(3s)).$$

The final vector of the quasi frame is defined by

$$B_q(s) = C_q(s) \wedge T_q(s) \wedge N_q(s),$$

which yields

$$B_q(s) = \frac{1}{\sqrt{10}}(3\sin s, -3\cos s, -\cos(3s), -\sin(3s)).$$

The quasi curvatures are calculated as

$$k_1 = -\frac{1}{10s}, k_2 = \frac{6}{5\sqrt{10}s}, k_3 = 0, k_4 = 0.$$

The parametrization of the ruled hypersurface generated by $T_q(s)$and $C_q(s)$is

$$\mathcal{H}(s, v, w) = \beta(s) + vT_q(s) + wC_q(s)$$

$$
= \begin{bmatrix} -s\sin\ s - \cos\ s \\ s\cos\ s - \sin\ s \\ -s\sin\,(3s) + \frac{1}{3}\cos\,(3s) \\ s\cos\,(3s) + \frac{1}{3}\sin\,(3s) \end{bmatrix} +
$$

$$
\frac{v}{\sqrt{10}} \begin{bmatrix} -\cos\ s \\ -\sin\ s \\ -3\cos\,(3s) \\ -3\sin\,(3s) \end{bmatrix} + w \begin{bmatrix} 0 \\ 0 \\ \cos\,(3s) \\ \sin\,(3s) \end{bmatrix}
$$

$$
=
$$

$$
\begin{bmatrix} -s\sin\ s - \cos\ s - \frac{v}{\sqrt{10}}\cos\ s \\ s\cos\ s - \sin\ s - \frac{v}{\sqrt{10}}\sin\ s \\ -s\sin\,(3s) + \frac{1}{3}\cos\,(3s) + \left(w - \frac{3v}{\sqrt{10}}\right)\cos\,(3s) \\ s\cos\,(3s) + \frac{1}{3}\sin\,(3s) + \left(w - \frac{3v}{\sqrt{10}}\right)\sin\,(3s) \end{bmatrix}
$$

The first-order partial derivatives with respect to s, v, w are

$$
\mathcal{H}_s = \begin{bmatrix} -s\cos\ s + \frac{v}{\sqrt{10}}\sin\ s \\ -s\sin\ s - \frac{v}{\sqrt{10}}\cos\ s \\ -3s\cos\,(3s) + \left(\frac{9v}{\sqrt{10}} - 3w\right)\sin\,(3s) \\ -3s\sin\,(3s) - \left(\frac{9v}{\sqrt{10}} - 3w\right)\cos\,(3s) \end{bmatrix},
$$

$$
\mathcal{H}_v = T_q(s) = \frac{1}{\sqrt{10}}(-\cos s\,,\,-\sin s\,,\,-3\cos(3s)\,,\,-3\sin(3s)),
$$

$$
\mathcal{H}_w = C_q(s) = (0, 0,\ \cos(3s)\,,\ \sin(3s)).
$$

The coefficients of the first fundamental form are

$$
a_{11} = \langle \mathcal{H}_s, \mathcal{H}_s \rangle = 10s^2 + \frac{v^2}{10} + 9w^2
$$

$$
a_{22} = \langle \mathcal{H}_v, \mathcal{H}_v \rangle = 1
$$

$$
a_{33} = \langle \mathcal{H}_w, \mathcal{H}_w \rangle = 1
$$

and

$$
a_{12} = a_{21} = \langle \mathcal{H}_s, \mathcal{H}_v \rangle = 0
$$

$$
a_{13} = a_{31} = \langle \mathcal{H}_s, \mathcal{H}_w \rangle = 0
$$

$$
a_{23} = a_{32} = \langle \mathcal{H}_v, \mathcal{H}_w \rangle = 0.
$$

The first fundamental form is calculated as

$$I=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=\begin{bmatrix}10s^2+\frac{v^2}{10}+9w^2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

The normal vector of hypersurface is

$$N=\frac{1}{\sqrt{10}}(3\sin\ s,-3\cos\ s,-\cos\ (3s),-\sin\ (3s)).$$

The second fundamental form is calculated as

$$II=\begin{bmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{bmatrix}=\begin{bmatrix}-\frac{6}{\sqrt{10}} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

by using the second-order partial derivatives

$$H_{ss}=\beta''(s)+v\,T''(s)+w\,C''(s)$$
$$H_{vv}=0$$
$$H_{ww}=0$$

and

$$H_{sv}=H_{vs}\quad =T'(s)$$
$$H_{sw}=H_{ws}\quad =C'(s)$$
$$H_{vw}=H_{wv}\quad =0$$

such that

$$b_{11}=<\mathcal{H}_{ss},N>=-\frac{6}{\sqrt{10}}s$$
$$b_{22}=<\mathcal{H}_{vv},N>=0$$
$$b_{33}=<\mathcal{H}_{ww},N>=0$$

and

$$b_{12}=b_{21}=<\mathcal{H}_{sv},N>=0$$
$$b_{13}=b_{31}=<\mathcal{H}_{sw},N>=0$$
$$b_{23}=b_{32}=<\mathcal{H}_{vw},N>=0.$$

The shape operator is defined by

$$S=\begin{bmatrix}\frac{-6\sqrt{10}s}{100s^2+v^2+90w^2} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Since the shape operator is diagonal matrix, the principal curvatures are given by the eigenvalues of the shape operator, which are given

$$\kappa_1 = \frac{-6\sqrt{10}s}{100s^2 + v^2 + 90w^2}, \kappa_2 = 0, \kappa_3 = 0.$$

Gauss and mean curvatures of the hypersurface are found as

$$K = \det(S) = 0$$

and

$$H = \frac{1}{3}\operatorname{tr}(S) = \frac{1}{3}(\kappa_1 + \kappa_2 + \kappa_3) = \frac{-2\sqrt{10}s}{100s^2 + v^2 + 90w^2}$$

respectively.

Corollary 3.5. The hypersurface generated by $\beta(s)$is flat.

Proof Since Gaussian curvature $K = \det(S) = 0$, it is obvious.

Corollary 3.6. The hypersurface generated by $\beta(s)$ is not globally minimal.

Proof Mean curvature of the hypersurface is found as

$$H = \frac{1}{3}\operatorname{tr}(S) = \frac{1}{3}(\kappa_1 + \kappa_2 + \kappa_3) = \frac{-2\sqrt{10}s}{100s^2 + v^2 + 90w^2}.$$

Therefore one can say, this hypersurface is not globally minimal since mean curvature Hdoes not vanish except for $s = 0$.

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Chapter 4

Plant Names in the *Kitâb-ı Maʿcûn* and Their Current Latin Equivalents in Binomial Nomenclature

Celalettin PERU[1], Harun ŞAHİN[2], Yusuf Kağan ALATAŞ[3] Murat ÜNLÜ[4], Mevlüt ALATAŞ[5]

1. INTRODUCTION

From the emergence of humankind to the present day, numerous treatment methods have been developed for both humans and animals. One such method involves the use of various plants to cure diseases. The science of herbal medicine, which arose from the fundamental need for survival, gradually evolved through works written in different languages across various geographies. These works spread widely as they were translated into the languages of different communities (Küçüker and Yıldız, 2018).

This medical dissemination can be observed in 45 surviving texts from the pre-Islamic Uyghur Turks to today. These texts are significant in the fields of medicine, pharmacology, and linguistics due to the medical and botanical terminology they contain (Bayat, 2016).

One of the works that provides prescriptions for treating different diseases is *Kitâb-ı Maʿcûn*, the focus of this study. No information exists regarding the author, the date of composition, or the date of transcription of this manuscript, which is written in 13 lines of Naskh script. The work, preserved in the Kastamonu Provincial Public Library in Türkiye, includes not only recipes for treating ailments but also terminology related to medicine, folk medicine concepts, disease and plant names, and linguistic features indicative of the period in which it was written.

[1]Dr., Metroport Business Center, Bahçelievler, İstanbul.
[2]Doç. Dr., Gazi University, Gazi Faculty of Education, Department of Turkish and Social Sciences Education, harunsahin75@gmail.com, Orcid No: 0000-0001-7178-7267
[3]Student, Yüksek Ihtisas University, Faculty of Medicine, Ankara.
[4]Dr., Wesia Kaya Thermal International Clinic, İzmir.
[5]Prof. Dr., Munzur University, Tunceli Vocational School, Department of Plant and Animal Production, mevlutalatas@gmail.com, Orcid No: 0000-0003-0862-0258

Reflecting the plain and simple style of Old Anatolian Turkish medical texts, *Kitâb-ı Maʿcûn* primarily mentions plant names used in ancient medical science or folk medicine (Aytaç, 2019). The plants mentioned in the text are known only within the region where the manuscript is found or its surroundings, often under local names that differ across geographies. For these plants regarded as curative in treating diseases to be used universally, studies are needed that identify their current Latin equivalents according to globally accepted binomial nomenclature.

This study aims to determine the plant names in *Kitâb-ı Maʿcûn* and present their corresponding modern Latin names in systematic botany. It is also the first research that examines this text from the perspective of plant systematics.

2. MATERIALS AND METHODS

The primary material of this study is the medical manuscript *Kitâb-ı Maʿcûn*. First, the plant names mentioned in the text were identified. These names appear in bold at the beginning of the first row in each section of the table. They are followed by information about the language of origin and explanatory or descriptive notes regarding each plant. Various dictionaries and sources were consulted while preparing these explanations (Johnson, 1952; Develioğlu, 2000; Unat et al., 2004; Önler, 2004, 2006; Bilgin, 2006; Şahin, 2007; Paçacıoğlu, 2014; Baytop, 2015; Baytar, 2017; Küçüker and Yıldız, 2018; Aytaç, 2019; Gümüşatam, 2022; Çakıcı, 2023).

For the plants with sufficient descriptive information, the corresponding taxa were identified at the species level in accordance with current binomial nomenclature. Those lacking sufficient detail were named at the genus level. Generally mentioned species that could not be classified systematically were left without taxonomic assignment. Some plant names or terms could not be located in dictionaries.

The APG III system (Haston et al., 2009), List of Turkish Plants (Vascular Plants) (Güner et al., 2012), Illustrated Flora of Turkey Vol. 1-2 (Güner et al., 2014, 2018), as well as the Bizimbitkiler (2013), IPNI (2024), and POWO (2024) databases were used to evaluate the validity and synonymy status of the Latin names.

3. RESULTS, AND DISCUSSION

The analysis of *Kitâb-ı Maʿcûn* revealed that 74 plant names are mentioned in the text. Written in Old Anatolian Turkish, the manuscript identifies these plants by their local names, and this study provides their modern Latin equivalents based on systematic botany.

Of these plants, **60 taxa** contained sufficient descriptive information to be identified at the species or subspecies level, **11 taxa** were identified at the genus level, and **three taxa**, which lacked sufficient detail for taxonomic determination, were not assigned to any systematic category (Table 1).

Table 1. Plant names in *Kitâb-ı Maʿcûn* and their current Latin equivalents in binomial nomenclature.

anīsūn: Greek. Anise. A herbaceous plant whose fruits are fragrant and carminative. ***Pimpinella anisum*** L.
anẕurūt: Persian. A dwarf tree that grows in warm countries, and its resin is used in wound treatment. ***Astragalus sarcocolla*** Dymock.
arpa otı: A type of grain, vetch. ***Hordeum vulgare*** L.
behmen-i beyāż: Latin. White behmen. White rabies herb, a thorny plant with a root resembling a radish or carrot, called kavza. ***Centaurea behen*** L.
belīlec: Persian. Belile herb. ***Terminalia bellirica*** (Gaertn.) Roxb. (*Bellerica myrobalan*)
bellūṭ: Arabic. Acorn. ***Quercus ithaburensis*** subsp. ***macrolepis*** (Kotschy) Hedge & Yalt.
besbāse: Arabic. White variety of harmala seed. ***Peganum harmala*** L.
beyān: Arabic. Licorice. A plant with purple flowers whose sweet roots are used medicinally. ***Glycyrrhiza glabra*** L.
bezr: Arabic. Linseed. ***Linum usitatissimum*** L.
birinç: Arabic. Rice. ***Oryza* sp.**
buġday: Wheat. ***Triticum aestivum*** L.
büber: Greek. Pepper. ***Capsicum annuum*** L.
cevz: Arabic. The walnut tree and its fruit that sheds its leaves in winter. ***Juglans regia*** L.
cevz-i bevvā: Arabic. Nutmeg tree. ***Myristica fragrans*** Houtt.
çam ṣaḳızı: Resin extracted from pine trees. ***Pinus* sp.**
çenār-ı tek: Persian. Sabin Juniper is a species belonging to the juniper genus from the cypress family. ***Juniperus sabina*** L.
çınṭıyāne: Greek. A mountain plant with yellow flowers. Gentian. ***Gentiana lutea*** L.

çörek otı: A herbaceous plant with blue flowers. ***Nigella sativa*** L.
dār-ı fülfül: Persian - Arabic. A long-grained spice, similar to black pepper, with a sharp flavor, used in medicine in the past. The unripe fruit of the long pepper tree. ***Piper* sp.**
dārçīnī/dārçīn: Persian. A species of evergreen tree from the laurel family. Cinnamon. ***Cinnamomum verum*** J.S. Presl.
defn toḫmı: Greek. The seed of the bay tree, which does not shed its leaves in winter and has a pleasant scent. ***Laurus nobilis*** L.
egri kesdāne: Latin. Galangal. A fragrant plant from the ginger family, used in medicine in the past. ***Alpinia officinarum*** Hance.
emlec: Arabic. Amlac herb, helile fruit. ***Phyllanthus emblica*** Linn.
enār ḳabı: Persian. Pomegranate peel. ***Punica granatum*** L.
erkek sürḫek: Persian. Male dogwood tree and its fruit. ***Cornus mas*** L.
ferfiyūn: Arabic. Euphorbia herb and medicinal glue obtained from it. ***Euphorbia* sp**.
fetrāśāliyūn: Greek. Wild celery. Macedonian parsley. ***Smyrnium connatum*** Boiss. & Kotschy
fınduḳ: Persian. Hazelnut. A small tree species and its fruit from the Bonito family. ***Corylus avellana*** L.
fülfül: Arabic. Pepper. Black pepper. ***Piper* sp.**
fülfül-i ebyeż: Arabic. White pepper. ***Capsicum annuum*** L.
fülfül-i esved: Arabic. Black pepper. ***Piper nigrum*** L.
günlük: Sweetgum tree. A sycamore-like tree and the gum obtained from it. ***Liquidambar orientalis*** Mill.
göz otı: Henbane. A poisonous plant whose leaves are used as a painkiller. Anzurut. ***Hyoscyamus niger*** L.
ḥabbü'ṣ-ṣanavber: Arabic. Stone pine. Pine nut cone grain. ***Pinus pinea*** L.
ḫaşḫāş: Arabic. A poisonous plant from the Papaveraceae family. Opium. ***Papaver somniferum*** L.
ḫavlincān: Persian. Galangal. A fragrant plant from the ginger family. ***Alpinia officinarum*** Hance.
ḥavvar-ı Hindī: Arabic. Indian poplar.
hevc: Persian. Carrot. ***Daucus carota*** L.
ḫuṣṣeü'ṡ-ṡaʿleb: Arabic. A plant called fox testicle.. (*Orchis hircina* (L.) Crantz)

Himantoglossum hircinum (L.) Spreng.
ısırgan dikeni: Nettle. ***Urtica dioica*** L.
kād-i Hindī: Persian. A plant brought from India with blood-thinning properties that is applied to wounds and circumcision sites.
ḳāḳūle: Arabic. A plant from the ginger family. ***Elettaria cardamomum*** (L.) Maton)
ḳāḳūle-i ṣaġīr/ ḳāḳūle-i ṣıġār: Arabic. Nutmeg. ***Myristica fragrans*** Houtt.
ḳaraca üzüm: Black grape. ***Vitis vinifera*** L.
ḳaranfīl: Persian. A herbaceous ornamental plant from the carnation family. ***Dianthus* sp.**
kebābe: Arabic. Indian pepper, tailed pepper. ***Piper cubeba*** L.
kerefs: Arabic. A fragrant plant whose roots and leaves are used as a vegetable. Celery. ***Apium graveolens*** L.
ḳızılboya: A plant with pale yellow flowers.
ḳızıl gül: Gallic rose. It is a species of flowering plant in the rose family. ***Rosa gallica*** L.
limon: Greek. A citrus tree and its fruit. ***Citrus limon*** (L.) Osbeck
lisān-ı ʿaṣāfir: Arabic. Rosemary is an evergreen plant that blooms pale blue flowers. ***Salvia rosmarinus*** Spenn.
mahmūde: Arabic. A creeping herbaceous plant with thick roots and pale yellow flowers. Its root and the milk obtained from its roots are used as a laxative. ***Convolvulus scammonia*** L.
māzū: Persian. A dwarf tree species from the cypress family that does not shed its leaves in winter and its fruit. ***Thuja* sp.**
merv: Arabic. Wild mint. ***Mentha pulegium*** L.
nīl-i Hindī: Arabic. Blue water lily. Indigo plant ***Isatis tinctoria*** L.
pırasa: Greek. A plant from the lily family. ***Allium ampeloprasum*** L.
rāziyāne: Persian. A herbaceous plant from the parsley family. Fennel. ***Foeniculum vulgare*** Mill.
saḳız: Gum tree, mastic. ***Pistacia lentiscus*** L.
sināmeki: Arabic. A legume plant with yellow flowers and a bush-like appearance, whose leaves are used as a laxative. ***Cassia* sp.**

siñirlüce otı/yaprağı: Plantain. Herbaceous plants with wound-healing leaves. ***Plantago major*** L.
ṣoġan: Onion. ***Allium cepa*** L.
sūrincān: Persian. Colchicum. Sorincan tree. ***Colchicum autumnale*** L.
sünbül-i Hindī: Persian. A type of hyacinth. Valerian plant. ***Nardostachys jatamansi*** (D.Don) DC.
şalġam: Persian. A tuberous plant from the cruciferous family. ***Brassica napus*** L., ***Brassica rapa*** L.
şırlaġan: Arabic. Sesame oil. ***Sesamum indicum*** L.
ṭopalaḳ: Incirop. A tuberous herbaceous plant with white flowers. ***Bunium ferulaceum*** Sm.
turb: Persian. A plant from the Brassicaceae family. Black radish. ***Raphanus raphanistrum*** subsp**.** ***sativus*** (L.) Schmalh.
ʿūdü'l- ḳahr: Arabic. Valerian plant. A plant with a strong odor and purple flowers. ***Valeriana*** **sp.**
yabān kerevizi: Lovage. A foul-smelling plant grown for its medicinal properties. ***Selinum alatum*** (M.Bieb.) Poir.
yabān reyḥānı: A type of basil. ***Ocimum*** **sp.**
zaʿferān: Arabic. Saffron. A tuberous plant that blooms purple flowers in autumn. ***Crocus sativus*** L.
zencebīl: Arabic. Ginger. A fragrant plant. ***Zingiber officinale*** Roscoe.
zerāvend: Persian. Dutchman's pipe. Herbaceous plants with heart-shaped leaves and pipe-shaped flowers. ***Aristolochia*** **sp.**
zerneb: Arabic. A herb that smells nice like orange. ***Taxus baccata*** L.

Modern medicine, which seeks alternatives to synthetic drugs, increasingly explores treatment approaches rooted in historical knowledge and preventive medicine (Gümüşatam, 2022). *Kitâb-ı Maʿcûn*, the focus of this study, is among the most significant herbal references handed down from the past. By identifying 74 plant names in the manuscript and providing their modern botanical equivalents, this study offers an important contribution to both medical science and folk medicine.

Furthermore, this research is particularly valuable as it represents the first systematic botanical analysis of *Kitâb-ı Maʿcûn*, thereby bridging traditional medical knowledge with contemporary plant systematics.

Presenting the scientific names of the plants mentioned in the text will support the broader use of this work not only within Turkish geography but across the world in contexts of healing and medicinal research.

Lastly, the study will serve as a resource for future work in Turkish history, Turkish literature, and modern and traditional medicine.

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